GLOBAL STABILITY OF THE POSITIVE SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS AND ASYMPTOTIC MEAN SQUARE STABILITY OF EQUILIBRIUM POINTS IF WITH STOCHASTIC PERTURBATIONS

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Abstract*: The main objective of this paper is to study the boundedness character, the periodic character, the convergence and the global stability of the positive solutions of the difference equation:*

$$u_{n+1} = \frac{A + \sum_{i=0}^{k} \alpha_i u_{n-i}}{B + \sum_{i=0}^{k} \beta_i u_{n-i}}, \ n \in Z=\{0,1,...,\}, \tag{1.1}$$

where A, B, α_i, β_i *and the initial conditions*

$$u_i = \varphi_i, \ i \in Z_0 = \{-k,-k+1,...,0\},$$

while k *is a positive integer number and the necessary and sufficient conditions for asymptotic mean square stability of the equilibrium point of fractional difference equation is exposed to stochastic perturbations* ξ_n *which are directly proportional to the deviation of the system state* u_n *from the equilibrium point* $\bar{u}$*, the form* $\sigma(u_n - \bar{u})\xi_{n+1}$.

Keywords and phrases: *Asymptotic mean square stability, global stability, nonlinear difference equations, periodic character, stochastic perturbations.*
2000 Mathematics Subject Classification**:** *Primary* 39A10, 39A11; *secondary* 93E15.

1. INTRODUCTION

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc [9]. The case where any of A, B, α_i, β_i is allowed to be zero gives different special cases of (1.1) which are studied by many authors (see, e.g, [1], [2], [3], [4], [12]). Furthermore, the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations. Note that the difference equation (1.1) has been extensively studied in the special case k = 1 in the monograph [6]. So, the results presented in our paper are new.

Definition 1.1. The equilibrium point $\bar{u}$ of the equation

$$u_{n+1} = f(u_n, u_{n-1}, ..., u_{n-k}), \quad n = 0,1,...$$

is the point that satisfies the condition:

$$\bar{u} = f(\bar{u}, \bar{u}, \ldots, \bar{u}).$$

Definition 1.2. The equilibrium point $\bar{u}$ of equation (3) is said to be:

1. locally stable, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $\{u_n\}$ with initial conditions

$$u_{-k}, u_{-k+1} \ldots, u_0 \in (\bar{u} - \delta, \bar{u} + \delta),$$

we have $|u_n - \bar{u}| < \varepsilon$, for all $n \in N$.

2. locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that for any initial conditions

$u_{-k}, u_{-k+1}\ldots, u_0 \in (\bar{u}-\gamma, \bar{u}+\gamma)$,
the corresponding solution $\{u_n\}$ tends to $\bar{u}$.

3. **global attractor** if every solution $\{u_n\}$ converges to $\bar{u}$ as $n \to \infty$.

4. **globally asymptotically stable** if $\bar{u}$ is locally asymptotically stable and $\bar{u}$ is also global attractor.

5. **unstable** if $\bar{u}$ is not locally stable.

Definition 1.3. A sequence $\{u_n\}$, $n \geq -k$ is said to be periodic with period p if $u_{n+p} = u_n$ for all $n \geq -k$. A sequence $\{u_n\}$, $n \geq -k$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Assume that

$$\begin{cases} \tilde{a}_j = \sum_{i=j}^{k} \alpha_i,\ \tilde{b}_j = \sum_{i=j}^{k} \beta_i, \quad j=0,1,...,k \\ \tilde{a} = \tilde{a}_0,\ \tilde{b} = \tilde{b}_0, \\ \bar{a} = \sum_{i=0}^{k} (-1)^i \alpha_i,\ \bar{b} = \sum_{i=0}^{k} (-1)^i \beta_i \end{cases} \quad (1.2)$$

and suppose that equation (1.1) has some point of equilibrium $\bar{u}$ (not necessary a positive one).

Then by assumption

$$B + \tilde{b}\,\bar{u} \neq 0 \quad (1.3)$$

the equilibrium point $\bar{u}$ is defined by the algebraic equation:

$$\bar{u} = \left(A + \tilde{a}\bar{u}\right) / \left(B + \tilde{b}\bar{u}\right) \quad (1.4)$$

By condition (1.3) equation (1.4) can be transformed to the form:

$$\tilde{b}\,\bar{u}^2 - (\tilde{a} - B)\,\bar{u} - A = 0 \quad (1.5)$$

It is clear that if

$$(\tilde{a} - B)^2 + 4A\tilde{b} > 0 \quad (1.6)$$

equation (1.1) has two points of equilibrium:

$$\bar{u}_1 = \frac{\tilde{a} - B + \sqrt{(\tilde{a}-B)^2 + 4A\tilde{b}}}{2\tilde{b}} \quad (1.7)$$

and

$$\bar{u}_2 = \frac{\tilde{a} - B - \sqrt{(\tilde{a}-B)^2 + 4A\tilde{b}}}{2\tilde{b}} \quad (1.8)$$

If

$$(\tilde{a} - B)^2 + 4A\tilde{b} = 0 \quad (1.9)$$

then equation (1.1) has only one point of equilibrium:

$$\bar{u} = (\tilde{a} - B) / 2\tilde{b} \quad (1.10)$$

And at last if

$$(\tilde{a} - B)^2 + 4A\tilde{b} < 0 \quad (1.11)$$

then equation (1.1) has not equilibrium points.

Remark 1.1. Consider the case $A=0, \tilde{b} \neq 0$. From (1.4) we obtain the following. If, $B \neq 0$ and $\tilde{a} \neq B$, then equation (1.1) has two points of equilibrium:

$$\bar{u}_1 = \frac{\tilde{a} - B}{\tilde{b}}, \quad \bar{u}_2 = 0 \quad (1.12)$$

If, $B \neq 0$ and $\tilde{a} = B$¸ then equation (1.1) has only one point of equilibrium: $\bar{u}=0$. If $B=0$¸ then equation (1.1) has only one point of equilibrium: $\bar{u} = \tilde{a}/\tilde{b}$.

Remark 1.2. Consider the case $A = \tilde{b} = 0$ and $B \neq 0$. If $\tilde{a} \neq B$ ¸ then equation (1.1) has only one point of equilibrium: $\bar{u}=0$. If $\tilde{a}=B$¸ then each solution $\bar{u}=$ const is an equilibrium point of equation (1.1). Consequently, the positive equilibrium point $\bar{u}$ of the difference equation (1.1) is given by (1.7).

Let f: $(0, \infty)^{k+1} \to (0, \infty)$ be a continuous function defined by

$$f(u_0, u_1, \ldots, u_k) = \frac{A + \sum_{i=0}^{k} \alpha_i u_i}{B + \sum_{i=0}^{k} \beta_i u_i} \quad (1.13)$$

The linearized equation associated of equation (1.1) about the positive equilibrium point $\bar{u}$ is:

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}(\bar{u}, \bar{u}, \ldots, \bar{u}) z_{n-i},\ n = 0,1,\ldots \quad (1.14)$$

or

$$z_{n+1} + \sum_{i=0}^{k} b_i z_{n-i} = 0 \quad (1.15)$$

Where

$$b_i = -\frac{\partial f}{\partial u_i}(\bar{u}, \bar{u}, \ldots, \bar{u}) = \frac{\beta_i \bar{u} - \alpha_i}{B + \tilde{b}\bar{u}} \quad (1.16)$$

Theorem 1.1 (see [1,7,9]). Assume that $a, b \in R$ and $k \in \{0, 1, 2, \ldots\}$. Then

$$|a| + |b| < 1 \quad (1.17)$$

is a sufficient condition for the asymptotic stability of the difference equation

$$u_{n+1}+au_n+bu_{n-k}=0,\ n=0,1,2,..., \quad (1.18)$$

Theorem 1.1 can be easily extended to a general linear difference equation.

Theorem 1.2. (see [1,7]). Let

$$u_{n+k}+p_1u_{n+k-1}+...+p_ku_n=0 \quad (1.19)$$
$$n=0,1,2,...,$$

where $p_1,p_2,...,p_k \in R$ and $k \in \{1,2,...\}$.

Then equation (1.19) is asymptotically stable provided that

$$\sum_{i=0}^{k}|p_i|<1 \quad (1.20)$$

2. MAIN RESULTS

In this section, we establish some results which show that the positive equilibrium point $\bar{u}$ of the difference equation (1.1) is globally asymptotically stable and every positive solution of the difference equation (1.1) is bounded, the periodic character and the necessary and sufficient conditions for asymptotic mean square stability of the equilibrium point of rational difference quation (1.1), if is exposed to stochastic perturbations ξ_n which are directly proportional to the deviation of the system state u_n from the equilibrium point $\bar{u}$, the form $\sigma(u_n-\bar{u})\xi_{n+1}$.

Theorem 2.1. Assume that $B>\tilde{a}$ holds. Let $\{u_n\}_{n=-k}^{\infty}$ be a solution of the difference equation (1.1) such that for some $n_0 \ge 0$, either

$$u_n \ge \bar{u} \text{ for } n \ge n_0 \quad (2.1)$$
$$u_n \le \bar{u} \text{ for } n \ge n_0 \quad (2.2)$$

Then u_n converges to $\bar{u}$ as $n\to\infty$, that is,

$$\lim_{n\to\infty} u_n=\bar{u} \quad (2.3)$$

Proof. Assume that (2.1) holds. The case where (2.2) holds is similar and will be omitted. Then, for $n \ge n_0+k$, we deduce that

$$u_{n+1}=\frac{A+\sum_{i=0}^{k}\alpha_iu_{n-i}}{B+\sum_{i=0}^{k}\beta_iu_{n-i}}=\left[\sum_{i=0}^{k}\alpha_iu_{n-i}\right]\left[\frac{1+A\Big/\sum_{i=0}^{k}\alpha_iu_{n-i}}{B+\sum_{i=0}^{k}\beta_i u_{n-i}}\right]\le$$

$$\le\left[\sum_{i=0}^{k}\alpha_iu_{n-i}\right]\left[\frac{1+A/\tilde{a}\,\bar{u}}{B+\tilde{b}\,\bar{u}}\right]=$$

$$=\left[\sum_{i=0}^{k}\alpha_iu_{n-i}\right]\left[\frac{A+\tilde{a}\,\bar{u}}{\tilde{a}\,\bar{u}\,(B+\tilde{b}\,\bar{u})}\right] \quad (2.4)$$

With the aid of (1.3), the last inequality becomes:

$$u_{n+1}\le\sum_{i=0}^{k}\alpha_iu_{n-i}\Big/\tilde{a} \quad (2.5)$$

and so

$$u_{n+1}\le\max_{0\le i\le k}\{u_{n-i}\} \text{ for } n\ge n_0+k \quad (2.6)$$

Set

$$v_n=\max_{0\le i\le k}\{u_{n-i}\} \text{ for } n\ge n_0+k \quad (2.7)$$

Then clearly

$$v_n\ge u_{n+1}\ge\bar{u} \quad \text{for } n\ge n_0+k \quad (2.8)$$

Next, we claim that

$$v_{n+1}\ge v_n \quad \text{for } n\ge n_0+k \quad (2.9)$$

Now, we have

$$v_{n+1}=\max_{0\le i\le k}\{u_{n+1-i}\}=\max\{u_{n+1},\ \max_{0\le i\le k}\{u_{n-i}\}\}\le$$

$$\le\max\{u_{n+1},v_n\}=v_n \quad (2.10)$$

From (2.8) and (2.9), it follows that the sequence $\{v_n\}$ is convergent and that

$$v=\lim_{n\to\infty}v_n\ge\tilde{u} \quad (2.11)$$

Furthermore, we get

$$u_{n+1}\le\frac{A+\sum_{i=0}^{k}\alpha_iu_{n-i}}{B+\tilde{b}\,\bar{u}}\le\frac{A+\tilde{a}\,v_n}{B+\tilde{b}\,\bar{u}} \quad (2.12)$$

From this and by using (2.9) we obtain,

$$u_{n+1}\le\frac{A+\tilde{a}\,v_{n+i-1}}{B+\tilde{b}\,\bar{u}}\le\frac{A+\tilde{a}\,v_n}{B+\tilde{b}\,\bar{u}} \quad (2.13)$$

for $i=1,\dots,k+1$.

Then

$$v_{n+k+1}=\max_{0\le i\le k+1}\{u_{n+i}\}\le u_{n+1}\le\frac{A+\tilde{a}\,v_n}{B+\tilde{b}\,\bar{u}} \quad (2.14)$$

and by letting $n\to\infty$, we obtain

$$v\le\frac{A+\tilde{a}\,v}{B+\tilde{b}\,\bar{u}} \quad (2.15)$$

Consequently, we obtain

$$v\left(1-\frac{\tilde{a}}{B+\tilde{b}\bar{u}}\right)\le\frac{A}{B+\tilde{b}\bar{u}} \quad (2.16)$$

From (1.3) and (2.16), we deduce that $v\le\bar{u}$, and in view of (2.11), we obtain $v=\bar{u}$. Thus, the proof of Theorem 2.1 is completed.

Theorem 2.2. Let $\{u_n\}_{n=-k}^{\infty}$ be a positive solution of the difference equation (1.1) and B >1. Then there exist positive constants m and M such that

$$m \le u_n \le M,\ n=0,1,... \tag{2.17}$$

Proof. From the difference equation (1.1), we have, when B >1

$$u_{n+1} \le \frac{A}{B} + \frac{1}{B}\left(\sum_{i=0}^{k} \alpha_i u_{n-i}\right),\ n=0,1,... \tag{2.18}$$

Consider the linear difference equation

$$w_{n+1} = \frac{A}{B} + \frac{1}{B}\left(\sum_{i=0}^{k} \alpha_i w_{n-i}\right),\ n=0,1,... \tag{2.19}$$

with the initial conditions $w_i = u_i > 0$, $i=-k,\dots,-1,0$. It follows by complete induction that

$$u_n \le w_n \tag{2.20}$$

First of all, assume that $B \ge \tilde{a}$. Then we have $A/(B-\tilde{a})$ is a particular solution of (2.19) and every solution of the homogeneous equation which is associated with (2.19) tends to zero as $n\to\infty$. Hence

$$\lim_{n\to\infty} w_n = \frac{A}{B-\tilde{a}} \tag{2.21}$$

From this (19) and (2.20), it follows that the sequence $\{u_n\}$ is bounded from above by a positive constant M say. That is,

$$u_n \le M,\ n=0,1,... \tag{2.22}$$

Set

$$m = \frac{A}{B+\tilde{b}\,M} \tag{2.23}$$

then we have

$$u_{n+1} = \frac{A + \sum_{i=0}^{k} \alpha_i u_{n-i}}{B + \sum_{i=0}^{k} \beta_i u_{n-i}} \ge \frac{A}{B+\tilde{b}M} = m \tag{2.24}$$

and consequently, we get

$$m \le u_n \le M,\ n=0,1,... \tag{2.25}$$

which completes the proof of Theorem 2.2 when $B>\tilde{a}$.

Second, consider the case when $B \ge \tilde{a}$. It suffices to show that $\{u_n\}$ is bounded from above by some positive constant. For the sake of contradiction, assume that $\{u_n\}$ is unbounded. Then there exists a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty} n_j = \infty,\ \lim_{j\to\infty} u_{1+n_j} = \infty,\ u_{1+n_j} =$$
$$= \max\{u_n;\ -k \le n \le 1+n_j\},\ j=0,1,2,... \tag{2.26}$$

From (2.18), we deduce that

$$\sum_{i=0}^{k} \alpha_i u_{-i+n_j} \ge B u_{1+n_j} - A \tag{2.27}$$

Taking the limit as $j \to \infty$ of both sides of the last inequality, we obtain

$$\lim_{j\to\infty} \sum_{i=0}^{k} \alpha_i u_{-i+n_j} = \infty \tag{2.28}$$

It is easy enough to show that $u_{-i+n_j} \le u_{1+n_j}$, $(i=0,1,2,...k)$ and then as $\tilde{a} = \sum_{i=0}^{k} \alpha_i$ we have:

$$\sum_{j=0}^{k} \alpha_j u_{-i+n_j} \le \tilde{a} u_{1+n_j} \tag{2.29}$$

From the last inequality and the difference equation (1.1), we obtain:

$$0 \le \tilde{a} u_{1+n_j} - \sum_{i=0}^{k} \alpha_i u_{-i+n_j} =$$
$$= \frac{\tilde{a}A + \sum_{i=0}^{k} \alpha_i u_{-i+n_j}\left[\tilde{a} - B - \sum_{i=0}^{k} \beta_i u_{-i+n_j}\right]}{B + \sum_{i=0}^{k} \beta_i u_{-i+n_j}} \tag{2.30}$$

Consequently, it follows that

$$\sum_{i=0}^{k} \beta_i u_{-i+n_j} \le \tilde{a} - B \tag{2.31}$$

Then for every $i=0,1,2,\dots,k$ for which β_i is positive, the subsequence $\{u_{-i+n_j}\}$ is bounded which implies that the sequence $\left\{\sum_{i=0}^{k} \alpha_i u_{-i+n_j}\right\}$ is also bounded. This contradicts (2.28) and the proof of the Theorem 2.2 is completed.

Theorem 2.3. Assume that $B>\tilde{a}$ holds. Then the positive equilibrium point $\bar{u}$ of the difference equation (1.1) is globally asymptotically stable.

Proof. The linearized equation (1.15) with (1.16) can be written in the form

$$z_{n+1} + \sum_{i=0}^{k} \frac{\beta_i \bar{u} - \alpha_i}{B + \tilde{b}\bar{u}} z_{n-i} = 0 \tag{2.32}$$

As $B > \tilde{a}$, we get

$$\sum_{i=0}^{k}\left|\frac{\beta_i\bar{u}-\alpha_i}{B+\tilde{b}\bar{u}}\right| \le \frac{\tilde{a}+\tilde{b}\bar{u}}{B+\tilde{b}\bar{u}}<1 \qquad (2.33)$$

Thus, by Theorem 2.2, we deduce that the equilibrium point $\bar{u}$ of the difference equation (1.1) is locally asymptotically stable. It remains to prove that the equilibrium point $\bar{u}$ is a global attractor. To this end, set

$$I=\liminf_{n\to\infty} u_n \text{ and } S=\limsup_{n\to\infty} u_n,$$

which by Theorem 2.4 are positive numbers.

Then, from the difference equation (1.1), we see that

$$S \le \frac{A+\tilde{a}S}{B+\tilde{b}I}, \quad I \ge \frac{A+\tilde{a}I}{B+\tilde{b}S} \qquad (2.34)$$

Hence

$$A+(\tilde{a}-B)I \le \tilde{b}IS \le A+(\tilde{a}-B)S \qquad (2.35)$$

From which it follows that I = S. Thus, the proof of Theorem 2.3 is completed.

Theorem 2.4. The necessary and sufficient condition for the difference equation (1.1) to have positive prime period two solutions is that both inequalities

$$A(\tilde{b}-\bar{b})^2-(\tilde{a}+\bar{a})(\tilde{b}-\bar{b})(B+\bar{a})<\tilde{b}(B+\bar{a})^2 \qquad (2.36)$$

$$B+\bar{a}<0 \qquad (2.37)$$

are valid.

Proof. First, suppose that there exist positive prime period two solutions

$$\ldots,P,Q,P,Q,\ldots \qquad (2.38)$$

of the difference equation (1.1). We will prove that the condition (2.36) holds. It follows from the difference equation (1.1) that

$$\begin{cases} P=\dfrac{A+\alpha_0Q+\alpha_1P+\alpha_2Q+\alpha_3P+\ldots}{B+\beta_0Q+\beta_1P+\beta_2Q+\beta_3P+\ldots} \\ Q=\dfrac{A+\alpha_0P+\alpha_1Q+\alpha_2P+\alpha_3O+\ldots}{B+\beta_0P+\beta_1Q+\beta_2P+\beta_3Q+\ldots} \end{cases} \qquad (2.39)$$

Consequently, we obtain

$$A+\alpha_0Q+\alpha_1P+\alpha_2Q+\alpha_3P+\ldots= =BP+\beta_0PQ+\beta_1P^2+\beta_2PQ+\beta_3P^2+\ldots \qquad (2.40)$$

$$A+\alpha_0P+\alpha_1Q+\alpha_2P+\alpha_3O+\ldots= =BQ+\beta_0PQ+\beta_1Q^2+\beta_2PQ+\beta_3Q^2+\ldots \qquad (2.41)$$

By subtracting, we deduce after some reduction that

$$P+Q=\frac{-(B+\bar{a})}{\beta_1+\beta_3+\ldots} \qquad (2.42)$$

while by adding we obtain

$$PQ=\frac{A(\beta_1+\beta_3+\ldots)-(\alpha_0+\alpha_2+\ldots)(B+\bar{a})}{\bar{b}(\beta_1+\beta_3+\ldots)} \qquad (2.43)$$

where $B+\bar{a}<0$. Now, it is clear from (2.42) and (2.43) that P and Q are two positive distinct real roots of the quadratic equation

$$t^2-(P+Q)t+PQ=0 \qquad (2.44)$$

Thus, we deduce that

$$\left(\frac{-(B+\bar{a})}{\beta_1+\beta_3+\ldots}\right)^2 > >4\left(\frac{A(\beta_1+\beta_3+\ldots)-(\alpha_0+\alpha_2+\ldots)(B+\bar{a})}{\bar{b}(\beta_1+\beta_3+\ldots)}\right) \qquad (2.45)$$

From (2.45), we obtain

$$A(\tilde{b}-\bar{b})^2-(\tilde{a}+\bar{a})(\tilde{b}-\bar{b})(B+\bar{a})< <\tilde{b}(B+\bar{a})^2 \qquad (2.46)$$

and hence the condition (2.36) is valid.

Conversely, suppose that the condition (2.36) is valid. Then, we deduce immediately from (2.46) that the inequality (2.45) holds. Consequently, there exist two positive distinct real numbers P and Q such that

$$\begin{cases} P=\dfrac{-(B+\bar{a})}{2(\beta_1+\beta_3+\ldots)}-\dfrac{1}{2}\sqrt{T_1} \\ Q=\dfrac{-(B+\bar{a})}{2(\beta_1+\beta_3+\ldots)}+\dfrac{1}{2}\sqrt{T_1} \end{cases} \qquad (2.47)$$

where $T_1 > 0$ which is given by the formula

$$T_1=\left(\frac{-(B+\bar{a})}{\beta_1+\beta_3+\ldots}\right)^2 - -4\left(\frac{A(\beta_1+\beta_3+\ldots)-(\alpha_0+\alpha_2+\ldots)(B+\bar{a})}{\bar{b}(\beta_1+\beta_3+\ldots)}\right) \qquad (2.48)$$

Thus, P and Q represent two positive distinct real roots of the quadratic equation (2.44).

Now, we are going to prove that P and Q are positive prime period two solutions of the difference equation (1.1). To this end, we assume that

$$u_{-k}=P,\ u_{-k+1}=Q,\ldots,u_{-1}=Q,\ u_0=P \qquad (2.49)$$

We wish to show that

$$u_1=Q,\ u_2=Q \qquad (2.50)$$

To this end, we deduce from the difference equation (1.1) that

$$u_1=\frac{A+\alpha_0 u_0+\alpha_1 u_{-1}+\ldots+\alpha_k u_{-k}}{B+\beta_0 u_0+\beta_1 u_{-1}+\ldots+\beta_k u_{-k}}=$$

$$=\frac{A+P(\alpha_0+\alpha_2+\ldots)+Q(\alpha_1+\alpha_3+\ldots)}{B+P(\beta_0+\beta_2+\ldots)+Q(\beta_1+\beta_3+\ldots)} \quad (2.51)$$

Dividing the denominator and numerator of (2.51) by

$$-(B+\bar{a})\,(\beta_1+\beta_3+\ldots)$$

and using (2.47)-(2.48), we obtain

$$u_1=\frac{\dfrac{-2A(\beta_1+\beta_3+\ldots)}{B+\bar{a}}+(1+\sqrt{K_1})(\alpha_0+\alpha_2+\ldots)+(1-\sqrt{K_1})(\alpha_1+\alpha_3+\ldots)}{\dfrac{-2B(\beta_1+\beta_3+\ldots)}{B+\bar{a}}+(1+\sqrt{K_1})(\beta_0+\beta_2+\ldots)+(1-\sqrt{K_1})(\beta_1+\beta_3+\ldots)}=$$

$$=\frac{\tilde{a}-2A\ (\beta_1+\beta_3+\ldots)/(B+\bar{a})+\bar{a}\sqrt{K_1}}{\tilde{b}-2B\ (\beta_1+\beta_3+\ldots)/(B+\bar{a})+\bar{b}\sqrt{K_1}} \quad (2.52)$$

where

$$K_1=1-\frac{A(\tilde{b}-\bar{b})^2-(\tilde{a}+\bar{a})(\tilde{b}-\bar{b})(B+\bar{a})}{\bar{b}(B+\bar{a})^2} \quad (2.53)$$

and from the condition (2.36), we deduce that $K_1 > 0$. Multiplying the denominator and numerator of (2.52) by

$$\left(\tilde{b}-2B(\beta_1+\beta_3+\ldots)/(B+\bar{a})]\right)-\bar{b}\sqrt{K_1} \quad (2.54)$$

we have:

$$u_1=[\tilde{a}-2A(\beta_1+\beta_3+\ldots)/(B+\bar{a})]\times$$

$$\times\frac{[\tilde{b}-2B(\beta_1+\beta_3+\ldots)/(B+\bar{a})]-\bar{b}\,\bar{a}\sqrt{K_1}}{[\tilde{b}-2B(\beta_1+\beta_3+\ldots)/(B+\bar{a})]^2-\bar{b}^2K_1}+$$

$$+\frac{[\tilde{b}\,\bar{a}-\tilde{a}\,\bar{b}-\bar{a}\,\dfrac{2B(\beta_1+\beta_3+\ldots)}{B+\bar{a}}+\bar{b}\,\dfrac{2A(\beta_1+\beta_3+\ldots)}{B+\bar{a}}\sqrt{K_1}}{[\tilde{b}-2B(\beta_1+\beta_3+\ldots)/(B+\bar{a})]^2-\bar{b}^2K_1} \quad (2.55)$$

After some reduction, we deduce that

$$u_1=\frac{-(B+\bar{a})(1+\sqrt{K_1})}{2(\beta_1+\beta_3+\ldots)}\times$$

$$\times\frac{\left[2(\alpha_1+\ldots)(\beta_0+\ldots)-2(\alpha_0+\ldots)(\beta_1+\ldots)-\dfrac{2(\beta_1+\ldots)}{(B+\bar{a})(A\bar{b}-B\bar{a})}\right]}{\left[2(\alpha_1+\ldots)(\beta_0+\ldots)-2(\alpha_0+\ldots)(\beta_1+\ldots)-\dfrac{2(\beta_1+\ldots)}{(B+\bar{a})(A\bar{b}-B\bar{a})}\right]}=$$

$$=\frac{-(B+\bar{a})(1+\sqrt{K_1})}{2(\beta_1+\beta_3+\ldots)}=\frac{-(B+\bar{a})}{2(\beta_1+\beta_3+\ldots)}+\frac{1}{2}\sqrt{T_1}=Q \quad (2.56)$$

Similarly, we can show that

$$u_2=\frac{A+\alpha_0 u_1+\alpha_1 u_0+\ldots+\alpha_k u_{-(k-1)}}{B+\beta_0 u_1+\beta_1 u_0+\ldots+\beta_k u_{-(k-1)}}=$$

$$=\frac{A+Q(\alpha_0+\alpha_2+\ldots)+P(\alpha_1+\alpha_3+\ldots)}{B+Q(\beta_0+\beta_2+\ldots)+P(\beta_1+\beta_3+\ldots)}=P \quad (2.57)$$

By using the mathematical induction, we have

$$u_n=P,\ \ u_{n+1}=Q,\ (\forall)\, n\geq -k \quad (2.58)$$

Thus, the difference equation (1.1) has positive prime period two solutions

$$\ldots,P,Q,P,Q,\ldots \quad (2.59)$$

Hence the proof of Theorem 2.4 is completed.

Let now $\{\Omega,\sigma,P\}$ be a probability space and $\{F_i\in\sigma,\, i\in Z\}$ be a nondecreasing family of σ-algebras of σ, i.e. $F_{n_1}\subset F_{n_2}$ for $n_1 < n_2$, E be the expectation, $\xi_n,\ n\in Z$, be a sequence of F_n-adapted mutually independent random variables such that $E\xi_n=0$, $E\xi_n^2=1$. It is supposed that the rational difference equation (1.1) has an equilibrium point $\bar{u}$ and is exposed to additive stochastic perturbations type of $\sigma(u_n-\bar{u})\xi_{n+1}$ that are directly proportional to the deviation of the state u_n of system (1.1) from the equilibrium point $\bar{u}$. So, equation (1.1) takes the form

$$u_{n+1}=\frac{A+\sum_{i=0}^{k}\alpha_i u_{n-i}}{B+\sum_{i=0}^{k}\beta_i u_{n-i}}++\sigma(u_n-\bar{u})\xi_{n+1} \quad (2.60)$$

Note that the equilibrium point $\bar{u}$ of equation
(1.1) is also the equilibrium point of equation (2.60). Putting $v_n=u_n-\bar{u}$ we will center equation (2.60) in the neighborhood of the point of equilibrium $\bar{u}$. From (2.60) it follows that:

$$v_{n+1}=\frac{A+\sum_{i=0}^{k}(\tilde{a}_i-\tilde{b}_i\bar{u})v_{n-i}}{B+\tilde{b}\bar{u}+\sum_{i=0}^{k}\tilde{b}_i v_{n-i}}+\sigma v_n\xi_{n+1} \quad (2.61)$$

It is clear that stability of the trivial solution of equation (2.61) is equivalent to stability of the equilibrium point of equation (2.60).

Together with nonlinear equation (2.61) we will consider and its linear part

$$z_{n+1}=\sum_{i=0}^{k}\gamma_i z_{n-i}+\sigma z_n\xi_{n+1},\ \gamma_i=\frac{\tilde{a}_i-\tilde{b}_i\bar{u}}{B+\tilde{b}\bar{u}} \qquad (2.62)$$

Two following definitions for stability are used below.

Definition 2.1. The trivial solution of equation (2.61) is called stable in probability if for any $\varepsilon_1>0$ and $\varepsilon_2>0$ there exists $\delta>0$ such that the solution $v_n=v_n(\varphi)$ satisfies the condition

$$P\{\sup_{n\in Z}|v_n(\varphi)|>\varepsilon_1\}<\varepsilon_2$$

for any initial function φ such that

$$P\{\sup_{i\in Z_0}|v_n(\varphi_i)|\le\delta\}=1.$$

Definition 2.2. Zero solution of equation (2.62) is called mean square stable if for any $\varepsilon>0$ there exists $\delta>0$ such that the solution

$$z_n=z_n(\varphi)$$

satisfies the condition

$$E\,|z_n(\varphi)|^2<\varepsilon$$

for any initial function φ such that

$$\|\varphi\|^2=\sup_{i\in Z_0}|\varphi_i|^2<\delta.$$

If besides

$$\lim_{n\to\infty}E\,|z_n(\varphi)|^2=0$$

for any initial function φ then the trivial solution of equation (2.62) is called asymptotically mean square stable.

Since the order of nonlinearity of equation (2.61) is more than 1, then obtained stability conditions at the same time are ([9], [10]) conditions for stability in probability of the trivial solution of nonlinear equation (2.61) and therefore for stability in probability of the equilibrium point of equation (2.61).

Lemma 2.1. [4] If

$$\sum_{i=0}^{k}|\gamma_i|<\sqrt{1-\sigma^2} \qquad (2.63)$$

then the trivial solution of equation (2.62) is asymptotically mean square stable.

Put

$$\tilde{\beta}=\sum_{i=0}^{k}|\gamma_i|,\ \tilde{\alpha}=\sum_{i=1}^{k}|G_i|,\ G_i=\sum_{j=i}^{k}|\gamma_j| \qquad (2.64)$$

Lemma 2.2. [4] If

$$\tilde{\beta}^2+2\tilde{\alpha}\,|1-\tilde{\beta}|+\sigma^2<1 \qquad (2.65)$$

then the trivial solution of equation (2.62) is asymptotically mean square stable.

Lemma 2.3. [4] Let there exist the nonnegative functional

$$V_i=V(i,u_{-k},\dots,u_i),\ i\in Z$$

for which the conditions

$$EV(0,u_{-k},...,u_0)\le c_1\|\varphi\|^2,\ E\Delta V_i\le -c_2\,Eu_i^2,\ i\in Z$$

where

$$\Delta V_i=V_{i+1}-V_i,\ c_1>0,\ c_2>0$$

hold. Then equation (2.62) zero solution is asymptotic mean square stable.

Consider the vectors

$$\tilde{z}_n=(z_{n-k},\dots,z_{n-1},z_n)^t$$

and

$$b=(0,\dots,\sigma)^t$$

of dimension *k*+1 and the square matrix

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ \gamma_k & \gamma_{k-1} & \gamma_{k-2} & \cdots & \gamma_1 & \gamma_0 \end{pmatrix}$$

Then equation (2.62) can be described in the form

$$\tilde{z}_{n+1}=A\tilde{z}_n+bz_n\xi_{n+1} \qquad (2.66)$$

Let the square matrix $U=\|u_{i,j}\|$ of dimension k+1 has all zero elements except for

$$u_{k+1,k+1}=1$$

and consider the matrix equation

$$A'DA-D=-U \qquad (2.67)$$

Theorem 2.5. Let equation (2.67) have a positive semidefinite solution D with $d_{k+1,k+1}>0$. Then, for asymptotic mean square stability of equation (2.62) zero solution, it is necessary and sufficient that the inequality:

$$\sigma^2 d_{k+1,k+1}<1 \qquad (2.68)$$

hold.

Proof. Consider the functional

$$V_n = \tilde{z}'_n D \tilde{z}_n + \sigma^2 d_{k+1,k+1} \sum_{i=1}^{k} z^2_{n-i} \qquad (2.69)$$

Calculating $E\Delta V_i$ by virtue of (2.69), (2.66), we obtain:

$$E\Delta V_n =$$

$$= E\left[\tilde{z}'_{n+1} D \tilde{z}_{n+1} + \sigma^2 d_{k+1,k+1} \sum_{i=1}^{k} z^2_{n+1-i} - \tilde{z}'_n D \tilde{z}_n - \sigma^2 d_{k+1,k+1} \sum_{i=1}^{k} z^2_{n-i} \right] =$$

$$= E[(A\tilde{z}_n + b z_n \xi_{n+1})' D (A\tilde{z}_n + b z_n \xi_{n+1}) - \tilde{z}'_n D \tilde{z}_n] +$$

$$+ \sigma^2 d_{k+1,k+1} E(z_n^2 - z^2_{n-k}) =$$

$$= E[\tilde{z}'_n (A'D A) - D)\tilde{z}_n + b'D b z^2_{n-k}] +$$

$$+ \sigma^2 d_{k+1,k+1} E(z_n^2 - z^2_{n-k}) = (\sigma^2 d_{k+1,k+1} - 1) E z_n^2$$

Let conditions (2.68) hold. Then the functional (2.69) satisfies the conditions of Lemma 2.3. It means that equation (2.66) zero solution is asymptotic mean square stable. It follows that condition (2.68) is sufficient for asymptotic mean square stability of equation (2.66) zero solution. Let condition (2.68) not hold, i.e.,

$$d_{k+1,k+1} \ge 1 \ .$$

Then, $E\Delta V_i \ge 0$. From here it follows that

$$\sum_{i=0}^{k-1} E\Delta V_i = EV_k - EV_0 \ge 0$$

i.e., $EV_i \ge EV_0 > 0$.

It means that equation (2.62) zero solution cannot be mean square stable. Therefore, condition (2.68) is necessary for asymptotic mean square stability of equation (2.62) zero solution. Theorem is proved. Remark that for every *k*, equation (2.66) is the system of

$$(k+1)(k+2)/2$$

equations. Consider the different particular cases of equation (2.66).

Corollary 2.1. For $k = 1$ condition (2.68) takes the form

$$|\gamma_i| < 1, \ |\gamma_0| < 1 - \gamma_1 \qquad (2.70)$$

$$\sigma^2 < d_{22}^{-1} = \frac{(1-\gamma_1)((1-\gamma_1)^2 - \gamma_0^2)}{1-\gamma_1} \qquad (2.71)$$

If, in particular, $\sigma = 0$ then condition (2.68) is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (2.62) for k = 1.

Remark 2.1. Put $\sigma = 0$. If $\tilde{\beta} = 1$ then the trivial solution of equation (2.62) can be stable (for example, $z_{n+1} = z_n$ or $z_{n+1} = 0{,}5(z_n + z_{n-1})$), unstable (for example, $z_{n+1} = 2z_n - z_{n-1}$) but cannot be asymptotically stable. Really, it is easy to see that if $\tilde{\beta} \ge 1$ (in particular, $\tilde{\beta} = 1$) then sufficient conditions (2.63) and (2.65) do not hold. Moreover, necessary and sufficient (for k = 1) condition (2.68) does not hold too since if (2.68) holds then we obtain a contradiction

$$1 \le \tilde{\beta} = \gamma_0 + \gamma_1 \le |\gamma_0| + \gamma_1 < 1$$

Remark 2.2. As it follows from the Lemmas 2.1, 2.2, 2.3 and Theorem 2.5 at the same time are conditions for stability in probability of the equilibrium point of equation (2.60). From conditions (2.63), (2.65) it follows that $|\tilde{\beta}| < 1$.

Let us check if this condition can be true for each equilibrium point. Suppose at first that condition (1.6) holds. Then equation (2.60) has two points of equilibrium $\bar{u}_1$ and $\bar{u}_2 = 0$ defined by (1.7) and (1.8) accordingly.

Putting

$$S = \sqrt{(\tilde{a} - B)^2 + 4A\tilde{b}}$$

via (2.64), (2.62), (1.4) we obtain that responding $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are:

$$\begin{cases} \tilde{\beta}_1 = \dfrac{\tilde{a} - \tilde{b}\bar{u}_1}{B + \tilde{b}\bar{u}_1} = \dfrac{\tilde{a} - \frac{1}{2}(\tilde{a} - B + S)}{\tilde{a} + \frac{1}{2}(\tilde{a} - B + S)} = \dfrac{\tilde{a} + B - S)}{\tilde{a} + B + S)} \\ \tilde{\beta}_2 = \dfrac{\tilde{a} - \tilde{b}\bar{u}_2}{\lambda + \tilde{b}\bar{u}_2} = \dfrac{\tilde{a} - \frac{1}{2}(\tilde{a} - B - S)}{\tilde{a} + \frac{1}{2}(\tilde{a} - B - S)} = \dfrac{\tilde{a} + B + S)}{\tilde{a} + B - S)} \end{cases} \qquad (2.72)$$

So, $\tilde{\beta}_1 \tilde{\beta}_2 = 1$. It means that the condition $|\tilde{\beta}| < 1$ holds only for one from the equilibrium points $\bar{u}_1$ and $\bar{u}_2$.

Namely,

if $\tilde{a} + B > 0$ then $|\tilde{\beta}_1| < 1$,

if $\tilde{a} + B < 0$ then $|\tilde{\beta}_2| < 1$,

if $\tilde{a} + B = 0$ then $\tilde{\beta}_1 = \tilde{\beta}_2 = -1$.

In particular, if A=0 then via Remark 1.1 and (2.62) we have:

$\tilde{\beta}_1 = B\,\tilde{a}^{-1}$, $\tilde{\beta}_2 = B^{-1}\tilde{a}$

Therefore,

$$\left|\tilde{\beta}_1\right| < 1 \quad \text{if} \quad |B| < |\tilde{a}|$$

$$\left|\tilde{\beta}_2\right| < 1 \quad \text{if} \quad |B| > |\tilde{a}|,$$

$$\left|\tilde{\beta}_1\right| = \left|\tilde{\beta}_2\right| = 1 \quad \text{if} \quad |B| = |\tilde{a}|.$$

so, via Remark 2.1 we obtain: equilibrium points $\bar{u}_1$ and $\bar{u}_2$ can be stable concurrently only if corresponding $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are negative concurrently. Suppose now that condition (1.9) holds. Then equation (2.60) has only one point of equilibrium (1.10). From (2.64), (2.62), (1.4), (1.10) it follows that corresponding $\tilde{\beta}$ equals

$$\tilde{\beta} = \frac{\tilde{a} - \tilde{b}\bar{u}}{B + \tilde{b}\bar{u}} = \frac{\tilde{a} - \frac{1}{2}(\tilde{a} - B)}{B + \frac{1}{2}(\tilde{a} - B)} = \frac{\tilde{a} + B)}{B + \tilde{a}} = 1$$

As it follows from Remark 2.1 this point of equilibrium cannot be asymptotically stable.

Corollary 2.2. Let $\bar{u}$ be an equilibrium point of equation (2.60) such that

$$\sum_{i=0}^{k} \left|\alpha_i - \beta_i \bar{u}\right| < \left|B + \tilde{b}\bar{u}\right| \sqrt{1 - \sigma^2}, \quad \sigma^2 < 1 \qquad (2.73)$$

Then the equilibrium point $\bar{u}$ is stable in probability.

The proof follows from (2.62), Lemma 2.1 and Remark 2.2.

Theorem 2.6. Let $\bar{u}$ be an equilibrium point of equation (2.60) such that

$$\left|\tilde{a} - \bar{b}\,\bar{u}\right| < \left|B + \tilde{b}\,\bar{u}\right| \qquad (2.74)$$

$$2\sum_{i=0}^{k} \left|\tilde{a}_i - \tilde{b}_i \bar{u}\right| < \left|B + \tilde{a}\right| - \sigma^2 \frac{(B + \tilde{b}\bar{u})^2}{\left|B - \tilde{a} + 2\tilde{b}\bar{u}\right|} \qquad (2.75)$$

Then the equilibrium point $\bar{u}$ is stable in probability.

Proof. Via (1.4), (2.62), (2.64) we have:

$$\tilde{\alpha} = \left|B + \tilde{b}\bar{u}\right|^{-1} \sum_{i=0}^{k} \left|\tilde{a}_i - \tilde{b}_i \bar{u}\right|$$

$$\tilde{\beta} = \left(\tilde{a} - \tilde{b}\,\bar{u}\right) \Big/ \left(B + \tilde{b}\,\bar{u}\right)$$

Rewrite (2.2.65) in the form

$$2\,\tilde{\alpha} < 1 + \tilde{\beta} - \frac{\sigma^2}{1 - \tilde{\beta}}, \quad \left|\tilde{\beta}\right| < 1$$

and show that it holds. From (2.74) it follows that $\left|\tilde{\beta}\right| < 1$. Via $\left|\tilde{\beta}\right| < 1$ we have:

$$1 + \tilde{\beta} = 1 + \frac{\tilde{a} - \tilde{b}\bar{u}}{B + \tilde{b}\bar{u}} = \frac{B + \tilde{a}}{B + \tilde{b}\bar{u}} > 0$$

$$1 - \tilde{\beta} = 1 - \frac{\tilde{a} - \tilde{b}\bar{u}}{B + \tilde{b}\bar{u}} = \frac{B - \tilde{a} + 2\tilde{b}\bar{u}}{B + \tilde{b}\bar{u}} > 0$$

so,

$$2\sum_{i=0}^{k} \left|\tilde{a}_i - \tilde{b}_i \bar{u}\right| < \left|B + \tilde{b}\bar{u}\right| \left(\frac{B + \tilde{a}}{B + \tilde{b}\bar{u}} - \sigma^2 \frac{(B + \tilde{b}\bar{u})^2}{\left|B - \tilde{a} + 2\tilde{b}\bar{u}\right|} \right) =$$

$$= \left|B + \tilde{a}\right| - \sigma^2 \frac{(B + \tilde{b}\,\bar{u})^2}{\left|B - \tilde{a} + 2\,\tilde{b}\,\bar{u}\right|}$$

It means that the condition of Lemma 2.2 holds. Via Remark 2.2 the proof is completed.

Corollary 2.3. An equilibrium point $\bar{u}$ of the equation

$$u_{n+1} = \frac{\mu + \alpha_0 u_n + \alpha_1 u_{n-1}}{\lambda + \beta_0 u_n + \beta_1 u_{n-1}} + \sigma(u_n - \bar{u})\xi_{n+1} \qquad (2.76)$$

is stable in probability if and only if

$$\left|\alpha_1 - \beta_1 \bar{u}\right| < \left|B + \tilde{b}\bar{u}\right| \left|\alpha_0 - \beta_0 \bar{u}\right| <$$

$$< (B - \alpha_1 + (\beta_0 + 2\beta_1)\bar{u}\,\mathrm{sign}(B + \tilde{b}\,\bar{u}) \qquad (2.77)$$

$$\sigma^2 < (B + \alpha_0 - \alpha_1 + 2\beta_1 \bar{u}) \times$$

$$\times \frac{(B + \alpha_1 + \beta_0 \bar{u})(B - \tilde{a} + 2\tilde{b}\bar{u})}{(B - \alpha_1 + (\alpha_0 + 2\beta_1)\bar{u}(B + \tilde{b}\bar{u})^2} \qquad (2.78)$$

The proof follows from (2.62), (2.68), (2.69).

3. CONCLUSIONS

This study of the establish some results which show that the positive equilibrium point $\bar{u}$ of the difference equation (1.1) is globally asymptotically stable and every positive solution of the difference equation (1.1) is bounded, the periodic character and the necessary and sufficient conditions for asymptotic mean square stability of the equilibrium point of rational difference equation (1.1), if is exposed to stochastic perturbations ξ_n which are directly proportional to the deviation of the system state u_n from the equilibrium point $\overline{u}$, the form $\sigma(u_n - \overline{u})\xi_{n+1}$.

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