A BERWALD Γ_0 – LINEAR CONNECTION $B\Gamma_0$ IN THE RIEMANN-LAGRANGE GEOMETRY OF 1-JET SPACES

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***Abstract**: This paper introduces the notions of a nonlinear connection Γ and of a Γ-linear connection $\nabla\Gamma$ on the 1-jet space $J^1(T,M)$. A particular nonlinear connection Γ_0 and a Berwald Γ_0 -linear connection $B\Gamma_0$ are produced by a pair of semi-Riemannian metrics. The adapted components of the torsion and curvature d-tensors of our Berwald connection are described.*

***Key words**: 1-jet spaces, nonlinear connections, Γ-linear connections, a Berwald connection.*

1. INTRODUCTION

According to Olver's opinion expressed in the monograph [4] and in some private discussions, we emphasize that the 1-jet bundle represents the most convenient *space of configurations* for the study of quantum and classical field theories.

For that reason, many researchers studied the differential geometry of the 1-jet spaces, in the sense of d-connections, d-torsions and d-curvatures.

The geometrical approach from this paper follows the direction of development of the differential geometry of the 1-jet space $J^1(T,M)$ initiated by Asanov [1] and uses the geometrical methods from the theory of Lagrange spaces developed by Miron and Anastasiei [2].

It is important to note that this geometrical approach allows a clear exposition of the multi-time physical-mathematical concepts studied by Neagu [3] and offers many original ideas for the geometric dynamics of a PDEs systems, developed by Udrişte [5].

2. NONLINEAR CONNECTIONS

Let us consider the jet bundle of first order $J^1(T,M)\to T\times M$, whose local coordinates $(t^\alpha,\ x^i,\ x^i_\alpha)$, where $\alpha=\overline{1,\ p}$, $i=\overline{1,\ n}$, transform after the rules:

$$\tilde{t}^\alpha=\tilde{t}^\alpha(t^\beta),\quad \tilde{x}^i=\tilde{x}^i(x^j),$$

$$\tilde{x}^i_\alpha=\frac{\partial\tilde{x}^i}{\partial x^j}\frac{\partial t^\beta}{\partial\tilde{t}^\alpha}x^j_\beta \tag{1}$$

By definition, a pair of local functions $\Gamma=\left(M^{(i)}_{(\alpha)\beta},N^{(i)}_{(\alpha)j}\right)$, which transform after the rules:

$$\tilde{M}^{(j)}_{(\beta)\mu}\frac{\partial\tilde{t}^\mu}{\partial t^\alpha}=M^{(k)}_{(\gamma)\alpha}\frac{\partial\tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial\tilde{t}^\beta}-\frac{\partial\tilde{x}^j_\beta}{\partial t^\alpha}$$

$$\tilde{N}^{(j)}_{(\beta)k}\frac{\partial\tilde{x}^k}{\partial x^i}=N^{(k)}_{(\gamma)i}\frac{\partial\tilde{x}^j}{\partial x^k}\frac{\partial t^\gamma}{\partial\tilde{t}^\beta}-\frac{\partial\tilde{x}^j_\beta}{\partial x^i} \tag{2}$$

is called a *nonlinear connection* on the 1-jet bundle $E=J^1(T,M)$.

A nonlinear connection Γ on the 1-jet space E produces an *adapted basis* of vector fields:

$$\left\{\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}\right\} \subset X(E) \tag{3}$$

where

$$\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha} \frac{\partial}{\partial x^j_\beta}$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i} \frac{\partial}{\partial x^j_\beta} \tag{4}$$

This adapted basis is extremely convenient in the study of the differential geometry of 1-jet spaces because the transformation rules of its elements have a simple tensorial form:

$$\frac{\delta}{\delta t^\alpha} = \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \frac{\delta}{\delta \tilde{t}^\beta}$$

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}$$

$$\frac{\partial}{\partial x^i_\alpha} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} \frac{\partial}{\partial \tilde{x}^j_\beta} \tag{5}$$

In this context, the Lie algebra $X(E)$ of the vector fields on the 1-jet bundle E decomposes as the following direct sum:

$$X(E) = X(H_T) \oplus X(H_M) \oplus X(V)$$

where

$$X(H_T) = \text{Span}\left\{\frac{\delta}{\delta t^\alpha}\right\}$$

$$X(H_M) = \text{Span}\left\{\frac{\delta}{\delta x^i}\right\}$$

$$X(V) = \text{Span}\left\{\frac{\partial}{\partial x^i_\alpha}\right\} \tag{6}$$

As a consequence, any vector field $X \in X(E)$ on the 1-jet space E can be unique written in the form:

$$X = h_T X + h_M X + vX \tag{7}$$

where h_T, h_M and v are the *canonical projections* of the above decomposition of the Lie algebra X(E).

3. Γ-LINEAR CONNECTIONS

Let Γ be a nonlinear connection on the 1-jet space $E = J^1(T, M)$ and let h_T, h_M and v be the canonical projections of the decomposition of the Lie algebra of vector fields $X(E)$.

By definition, a linear connection:

$$\nabla : X(E) \times X(E) \to X(E) \tag{8}$$

having the properties

$$\nabla h_T = 0, \nabla h_M = 0 \text{ and } \nabla v = 0 \tag{9}$$

is called a Γ-*linear connection* on the jet space of first order E.

Using the preceding definition, it immediately follows that a Γ-linear connection ∇ on the jet bundle of first order is unique determined by a set of *nine* local functions, denoted by:

$$\nabla\Gamma = \Big(\bar{G}^\alpha_{\beta\gamma}, G^k_{i\gamma}, G^{(i)(\beta)}_{(\alpha)(j)\gamma}, \bar{L}^\alpha_{\beta j}, L^k_{ij}, L^{(i)(\beta)}_{(\alpha)(j)k}, \bar{C}^{\alpha(\gamma)}_{\beta(k)}, C^{j(\gamma)}_{i(k)}, C^{(i)(\beta)(\gamma)}_{(\alpha)(j)(k)}\Big) \tag{10}$$

which are defined by the relations:

$$\nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\delta}{\delta t^\beta} = \bar{G}^\alpha_{\beta\gamma} \frac{\delta}{\delta t^\alpha}$$

$$\nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\delta}{\delta x^i} = G^k_{i\gamma} \frac{\delta}{\delta x^k}$$

$$\nabla_{\frac{\delta}{\delta t^\gamma}} \frac{\partial}{\partial x^i_\beta} = G^{(k)(\beta)}_{(\alpha)(i)\gamma} \frac{\partial}{\partial x^k_\alpha}$$

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta t^\beta} = \bar{L}^\alpha_{\beta j} \frac{\delta}{\delta t^\alpha}$$

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = L^k_{ij} \frac{\delta}{\delta x^k}$$

$$\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^i_\beta} = L^{(k)(\beta)}_{(\alpha)(i)j} \frac{\partial}{\partial x^k_\alpha}$$

$$\nabla_{\frac{\partial}{\partial x^j_\gamma}} \frac{\delta}{\delta t^\beta} = \bar{C}^{\alpha(\gamma)}_{\beta(j)} \frac{\delta}{\delta t^\alpha}$$

$$\nabla_{\frac{\partial}{\partial x^j_\gamma}} \frac{\delta}{\delta x^i} = C^{k(\gamma)}_{i(j)} \frac{\delta}{\delta x^k}$$

$$\nabla_{\frac{\partial}{\partial x^j_\gamma}} \frac{\partial}{\partial x^i_\beta} = C^{(k)(\beta)(\gamma)}_{(\alpha)(i)(j)} \frac{\partial}{\partial x^k_\alpha} \tag{11}$$

Using the transformation laws of the elements of the adapted basis of vector fields, together with the properties of the Γ-linear connection ∇, we deduce that the those nine adapted components $\nabla\Gamma$ transform after the rules:

$$\overline{G}^{\delta}_{\alpha\beta} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\delta}} = \tilde{\tilde{G}}^{\varepsilon}_{\mu\gamma} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} \frac{\partial \tilde{t}^{\gamma}}{\partial t^{\beta}} + \frac{\partial^2 \tilde{t}^{\varepsilon}}{\partial t^{\alpha} \partial t^{\beta}}$$

$$G^{k}_{i\gamma} = \tilde{G}^{m}_{j\beta} \frac{\partial x^k}{\partial \tilde{x}^m} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial \tilde{t}^{\beta}}{\partial t^{\gamma}}$$

$$G^{(k)(\beta)}_{(\gamma)(i)\alpha} = \tilde{G}^{(p)(\eta)}_{(\varepsilon)(j)\mu} \frac{\partial x^k}{\partial \tilde{x}^p} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \frac{\partial \tilde{x}^j}{\partial x^i}$$

$$\frac{\partial t^{\beta}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} + \delta^k_i \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \frac{\partial^2 t^{\beta}}{\partial \tilde{t}^{\mu} \partial \tilde{t}^{\varepsilon}}$$

$$\overline{L}^{\gamma}_{\beta j} \frac{\partial x^j}{\partial \tilde{x}^l} = \tilde{\tilde{L}}^{\eta}_{\mu l} \frac{\partial t^{\gamma}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\beta}}$$

$$L^{m}_{ij} \frac{\partial \tilde{x}^r}{\partial x^m} = \tilde{L}^{r}_{pq} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^q}{\partial x^j} + \frac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j}$$

$$L^{(k)(\beta)}_{(\gamma)(i)j} = \tilde{L}^{(r)(\eta)}_{(\nu)(p)l} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^{\nu}}{\partial t^{\gamma}} \frac{\partial \tilde{x}^p}{\partial x^i}$$

$$\cdot \frac{\partial t^{\beta}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{x}^l}{\partial x^j} + \delta^{\beta}_{\gamma} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j}$$

$$\overline{C}^{\gamma(\alpha)}_{\beta(i)} = \tilde{\tilde{C}}^{\mu(\delta)}_{\varepsilon(j)} \frac{\partial t^{\gamma}}{\partial \tilde{t}^{\mu}} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\beta}} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\delta}}$$

$$C^{k(\alpha)}_{i(j)} = \tilde{C}^{s(\beta)}_{p(r)} \frac{\partial x^k}{\partial \tilde{x}^s} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^r}{\partial x^j} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\beta}}$$

$$C^{(k)(\beta)(\alpha)}_{(\gamma)(i)(j)} = \tilde{C}^{(r)(\mu)(\nu)}_{(\varepsilon)(p)(q)} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \cdot$$

$$\cdot \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial t^{\beta}}{\partial \tilde{t}^{\mu}} \frac{\partial \tilde{x}^q}{\partial x^j} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\nu}} \tag{12}$$

Finally, we point out that for a given Γ-linear connection ∇ characterized by nine adapted components $\nabla\Gamma$, we can compute the adapted components of its torsion and curvature d-tensors. In this direction, some strong results from the monograph [3] prove that the torsion d-tensor field **T** of $\nabla\Gamma$ is determined by *twelve* effective adapted components, while the curvature d-tensor field **R** of $\nabla\Gamma$ is determined by *eighteen* effective adapted components.

The expressions of all these effective adapted components are locally described in [3].

4. A BERWALD CONNECTION

Let us suppose now that $h_{\alpha\beta}(t)$ and $\varphi_{ij}(x)$, where $t = (t^{\gamma})$ and $x = (x^k)$, are semi-Riemannian metrics on the manifolds T and M and let us consider that $H^{\gamma}_{\alpha\beta}(t)$ and $\Gamma^k_{ij}(x)$ are the Christoffel symbols of these metrics. Then, taking into account the transformation laws of the Christoffel symbols, by direct computations, we deduce that the pair of local functions $\Gamma_0 = \left(\breve{M}^{(i)}_{(\alpha)\beta}, \breve{N}^{(i)}_{(\alpha)j}\right)$, where:

$$\breve{M}^{(i)}_{(\alpha)\beta} = -H^{\gamma}_{\alpha\beta} x^i_{\gamma}$$

$$\breve{N}^{(i)}_{(\alpha)j} = \Gamma^i_{jm} x^m_{\alpha} \tag{13}$$

represents a nonlinear connection on the 1-jet space $J^1(T, M)$.

The nonlinear connection Γ_0 is called the *canonical nonlinear connection produced by the metrics* $h_{\alpha\beta}(t)$ *and* $\varphi_{ij}(x)$.

Moreover, if we study the transformation laws of the following set of nine local functions, denoted by

$$B\Gamma_0 = \left(H^{\alpha}_{\beta\gamma}, 0, G^{(i)(\beta)}_{(\alpha)(j)\gamma}, 0, \Gamma^k_{ij}, L^{(i)(\beta)}_{(\alpha)(j)k}, 0, 0, 0\right) \tag{14}$$

where

$$G^{(i)(\beta)}_{(\alpha)(j)\gamma} = -\delta^i_j H^{\beta}_{\alpha\gamma}$$

$$L^{(i)(\beta)}_{(\alpha)(j)k} = \delta^{\beta}_{\alpha}\Gamma^{i}_{jk} \qquad (15)$$

then we deduce that the adapted components $B\Gamma_0$ represent a Γ_0-linear connection on the 1-jet space $J^1(T,M)$.

The Γ_0-linear connection $B\Gamma_0$ is called the *Berwald connection associated to the metrics* $h_{\alpha\beta}(t)$ *and* $\varphi_{ij}(x)$.

Now, particularizing some general results from the work [3], we can conclude that all adapted components of the torsion d-tensor $\mathbf{T}_0$ of our Berwald connection are zero, except:

$$R^{(m)}_{(\mu)\alpha\beta} = -H^{\gamma}_{\mu\alpha\beta}x^{m}_{\gamma}$$

$$R^{(m)}_{(\mu)ij} = r^{m}_{ijl}x^{l}_{\mu} \qquad (16)$$

where $H^{\gamma}_{\mu\alpha\beta}(t)$ and $r^{m}_{ijl}(x)$ are the classical curvature tensors of the metrics $h_{\alpha\beta}(t)$ and $\varphi_{ij}(x)$.

Also, all adapted components of the curvature d-tensor $\mathbf{R}_0$ of our Berwald connection are zero, except:

$$R^{\delta}_{\alpha\beta\gamma} = H^{\delta}_{\alpha\beta\gamma},\ R^{l}_{ijk} = r^{l}_{ijk} \qquad (17)$$

5. CONCLUSIONS

As final remarks, let us note that, in the particular case (T,h) = (R,δ), the canonical nonlinear connection Γ_0 naturally generalizes the canonical nonlinear connection produced by the spray $2G^i = \Gamma^i_{jk}y^jy^k$, while our Berwald connection is the natural generalization of the classical linear Berwald connection from the theory of Lagrange spaces [2], which is produced by the nonlinear connection $N^i_j = \Gamma^i_{jk}y^k$.

In conclusion, the Berwald Γ_0-linear connection $B\Gamma_0$ is a natural example of Γ-linear connection, which ensures us that our geometrical theory upon the Γ-linear connections on 1-jet spaces is a fertile and good one.

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