NEW CLASSES OF R -COMPLEX HERMITIAN FINSLER SPACES WITH *(α,β)*-METRICS

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Abstract: *The aim of this paper is to investigate three special* R *-complex Finsler spaces with (α,β)-metrics.We characterize Weyl metric,quadratic metric and another special (α,β)-metric in* R *-complex Finsler spaces conditions.Some properties of these metrics are demonstrated.Finally we came with some explicit examples.*

Keywords: R *-complex Finsler space,(α,β)-metrics*

1. PRELIMINARIES

The study of R -complex Finsler spaces is quite new.It has been initiated in[10] and it have been recently developed in[3],[4].[7].

In the paper [10], it was extended the well-known defnition of a complex Finsler space [1], reducing the scalars to λ∈ R . The outcome was a new class of Finsler space called the R - complex Finsler spaces [10].

In this section we keep the general setting from [3,10] and subsequently we recall only some needed notions.

An R- complex Finsler space is a pair *(M,F)*. where F is a continuous function $F : T'M \rightarrow R_+$ satisfying the conditions:

i) $L = F^2$ is smooth on $\widetilde{T'M} = T'M\backslash\{0\}$;

ii) $F(z,\eta) \geq 0$ the equality holds if and only if $\eta = 0$;

iii) $F(z,\lambda\eta,\bar{z},\overline{\lambda\eta}) = |\lambda|\, F(z,\eta,\bar{z},\bar{\eta})$; $\forall\lambda \in R$,

The fundamental function L of a R -complex Finsler space, induces the following tensors:

$$g_{ij} = \frac{\partial^2 L}{\partial\eta^i\partial\eta^j};\, g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j};\, g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j}$$

which satisfy interesting properties, obtained as consequences of the homogeneity condition iii)

$$\frac{\partial L}{\partial\eta^i}\eta^i + \frac{\partial L}{\partial\bar{\eta}^i}\bar{\eta}^i = 2L;\; g_{ij}\eta^i + g_{j\bar{i}}\bar{\eta}^i = \frac{\partial L}{\partial\eta^j}$$

$$2L = g_{ij}\eta^i\eta^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j$$

$$\frac{\partial g_{ik}}{\partial\eta^j}\eta^j + \frac{\partial g_{ik}}{\partial\bar{\eta}^j}\bar{\eta}^j = 0;\; \frac{\partial g_{i\bar{k}}}{\partial\eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial\bar{\eta}^j}\bar{\eta}^j = 0$$

Having an R - complex Finsler space, if we suppose that F satisfies the regularity conditions: $g_{i\bar{j}}$ is nondegenerated, (i.e., $\det(g_{i\bar{j}})\neq 0$, in any $u\in T'M$), and it defines a positive definite Levi-form for all $z\in M$, then such a class of spaces is called R - complex Hermitian Finsler space.

Consider the sections of the complexified tangent bundle of T′M. Let $VT'M\subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial\eta^k}\}$, and VT″M its conjugate.

The idea of complex nonlinear connection, briefly (c.n.c.), is an instrument in 'linearization' of the geometry of the manifold T′M. A (c.n.c.) is a supplementary complex subbundle to VT′M in T′(T′M), i.e. $T'(T'M)=HT'M\oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^i\frac{\partial}{\partial\eta^j}\right\}$, where $N_k^i(z,\eta)$ are the coefficients of the (c.n.c.). The pair $\{\delta_k = \frac{\delta}{\delta z^k}, \partial_k = \frac{\partial}{\partial\eta^k}\}$ will be called the adapted frame of the (c.n.c.).

A (c.n.c.) related only to the fundamental function of the R - complex Hermitian Finsler space (M,F), (called Chern-Finsler (c.n.c.)), has the following local coefficients:

$$N_k^i = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^k\partial\bar{\eta}^m} = g^{\bar{m}i}\left(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s\right)$$

Also, in a R - complex Hermitian Finsler space, we have recovered the Chern-Finsler connection, which is metrical, of (1,0)- type, and it is given by

$L_{jk}^{i} = g^{\bar{m}i}(\delta_j g_{k\bar{m}}); C_{jk}^{i} = g^{\bar{m}i}(\dot{\partial}_j g_{k\bar{m}});$

$L_{j\bar{k}}^{i} = C_{j\bar{k}}^{i} = 0$

where δ_j is the frame corresponding to the Chern-Finsler (c.n.c.).

2. R -COMPLEX FINSLER SPACE WITH WEYL METRIC

We consider $z \in M$, $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$. An R - complex Finsler space (M,F),with Weyl metric is a space where:

$L = F^2 = 2\alpha\beta$

$\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = Re\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j$

$\beta(z, \eta, \bar{z}, \bar{\eta}) = Re\{b_i\eta^i\}$

Proposition 2.1: *The invariants of this class of* R *-complex Finsler spaces are:*

$\rho_0 = \frac{\beta}{\alpha}; \rho_1 = \alpha; \rho_{-2} = \frac{-\beta}{2\alpha^3}; \rho_{-1} = \frac{1}{2\alpha}; \mu_0 = 0$

Proposition 2.2:*The metric tensor field of a* R *-complex Finsler spaces with (α,β)-metric: L(α,β)= 2αβ is given by:*

$g_{i\bar{j}} = \frac{\beta}{\alpha} a_{i\bar{j}} - \frac{\beta}{2\alpha^3} l_i l_{\bar{j}} + \frac{1}{2\alpha}(b_{\bar{j}} l_i + l_{\bar{j}} b_i)$

Or in the equivalent form:

$g_{i\bar{j}} = \frac{\beta}{\alpha} a_{i\bar{j}} - \frac{\beta}{\alpha^3} l_i l_{\bar{j}} - \frac{\alpha}{2\beta} b_{\bar{j}} b_i + \frac{1}{2\alpha\beta} \eta_i \eta_{\bar{j}}$

The next aim is to find the formulas for the determinant and the inverse of the tensor field $g_{i\bar{j}}$. The solution is obtained by the following Lemma like in [7], for an arbitrary non-singular Hermitian matrix $Q_{i\bar{j}}$.

Lemma: *Suppose:*

- *$(Q_{i\bar{j}})$is a non-singular n×n complex matrix with inverse $(Q^{i\bar{j}})$*
- *C_i and $C_{\bar{i}} = \bar{C}_i$, i=1,..,n, are complex numbers;*
- *$C^i = Q^{i\bar{j}} C_{\bar{j}}$ and its conjugates;*
- *$C^2 = C^i C_i = C^{\bar{i}} C_{\bar{i}}$; $H_{i\bar{j}} = Q_{i\bar{j}} \pm C_i C_{\bar{j}}$*

Then

$Det(H_{i\bar{j}})=(1 \pm C^2)det(Q_{i\bar{j}})$

Whenever $(1 \pm C^2) \neq 0$ the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is

$H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \pm C^2} C^i C^{\bar{j}}$

Proposition 2.3: *For the* R *- complex Hermitian Finsler space with the metric F= $\sqrt{2\alpha\beta}$ the determinant and the inverse of the fundamental metric tensor $g_{i\bar{j}}$ are given by*

i) $g^{i\bar{j}} = \frac{\alpha}{\beta} H^{\bar{j}i}$

ii) $det(H_{i\bar{j}}) = \frac{(2\beta^2 + \alpha^2 A)(\alpha^2 - \gamma)}{\alpha^2(2\beta^2 + B)} det(a_{i\bar{j}})$

Where

$H^{\bar{j}i} = a^{\bar{j}i} + Q\eta^i\bar{\eta}^j + \left(T + \frac{\alpha^2 P}{\sqrt{2BN}} + 2\alpha P\right) b^i b^{\bar{j}}$

$\eta^i\bar{\eta}^j + \left(T + \frac{\alpha^2 P}{\sqrt{2BN}} + 2\alpha P\right) b^i b^{\bar{j}} + \left(R + \frac{\alpha PM}{N}\right) b^i \bar{\eta}^j + \left(S b^{\bar{j}} + \frac{\alpha PM}{N}\right) b^{\bar{j}} \eta^i + \frac{\beta P}{\sqrt{2}\alpha^2 N} l^i l^{\bar{j}}$

$+ \left(\frac{P}{\sqrt{2}N} + \frac{\beta P}{\alpha}\right) l^i b^{\bar{j}} + \left(\frac{P}{\sqrt{2}N} + \frac{\beta P}{\alpha}\right) b^i l^{\bar{j}} + \frac{PMB}{\alpha N} l^i \bar{\eta}^j + \frac{PMB}{\alpha N} \eta^i l^{\bar{j}}$

Example 1:

We consider α as in [4], given by

$\alpha^2(z, \eta) = \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |< z, \eta >|^2)}{(1 + \varepsilon|z|^2)^2}$

defined over the disk $\Delta_r^n = \left\{z \in C^n \,||z| < r, r = \sqrt{\frac{1}{|\varepsilon|}}\right\}, \varepsilon < 0$. We set $\beta(z, \eta) = Re\frac{<z,\eta>}{(1+\varepsilon|z|^2)}$, where $b_i = \frac{\bar{z}^i}{(1+\varepsilon|z|^2)}$ and we obtain

$F_\varepsilon = \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |<z,\eta>|^2)}{(1+\varepsilon|z|^2)^2} \pm \left(Re\frac{<z,\eta>}{(1+\varepsilon|z|^2)}\right)^2$

3. A SPECIAL CLASS OF R -COMPLEX FINSLER SPACE WITH *(α,β)*-METRIC

Following the ideas from real case we shall introduce a new class of R - complex

Finsler metrics.We take

$L(\alpha, \beta) = F^2 = \frac{(\alpha + \beta)^4}{8}$

In order to study the R - complex Hermitian Finsler space with this metric, we suppose that $a_{ij} = 0$. Thus, only the tensor field $g_{i\bar{j}}$ is invertible.

Proposition 3.1:*The invariants of this class of* R *- complex Hermitian Finsler space are:*

$$\rho_0 = \frac{(\alpha+\beta)^3}{4\alpha}, \rho_1 = \frac{(\alpha+\beta)^3}{4}$$

$$\rho_{-2} = \frac{(\alpha+\beta)^2(4\alpha+\beta)}{8\alpha^3}, \rho_{-1} = \frac{3(\alpha+\beta)^2}{8\alpha}$$

$$\mu_0 = \frac{3(\alpha+\beta)^2}{8}$$

Next step is to go forward and we demonstrate :

Theorem 3.1:*The metric tensor field of an* R *-complex Hermittian Finsler space a with (α,β)-metric L(α,β)=*$\frac{(\alpha+\beta)^4}{8}$ *is given by:*

$$g_{i\bar{j}} = \frac{(\alpha+\beta)^3}{4\alpha} a_{i\bar{j}} + \frac{(\alpha+\beta)^2(4\alpha+\beta)}{8\alpha^3} l_i l_{\bar{j}} + \frac{3(\alpha+\beta)^2}{8} b_i b_{\bar{j}} + \frac{3(\alpha+\beta)^2}{8\alpha}\left(b_{\bar{j}} l_i + l_{\bar{j}} b_i\right)$$

Or in the equivalent form:

$$g_{i\bar{j}} = \frac{(\alpha+\beta)^3}{4\alpha} a_{i\bar{j}} + \frac{(\alpha+\beta)^3}{8\alpha^3} l_i l_{\bar{j}} + \frac{6}{(\alpha+\beta)^4} \eta_i \eta_{\bar{j}}$$

After some preparations we compute the inverse of the fundamental metric tensor:

Proposition 3.2: *For the* R *- complex Hermitian Finsler space with the metric L=*$F^2 = \frac{(\alpha+\beta)^4}{8}$ *the determinant and the inverse of the fundamental metric tensor* $g_{i\bar{j}}$ *are given by:*

$$i)\ H^{\bar{j}i} = a^{\bar{j}i} - \left(\frac{1}{2\alpha^2+\gamma} + P|M|^2\right)\eta^i\bar{\eta}^j - N^2 P b^i b^{\bar{j}} - MNPb^{\bar{j}}\eta^i - \bar{M}NPb^i\bar{\eta}^j$$

$$ii)\ det\left(H_{i\bar{j}}\right) = \left[\frac{(2\alpha^2+\gamma)}{2\alpha^2} - \frac{12|\mu|^2}{\alpha(\alpha+\beta)^7} + \frac{3(2\alpha^2+\gamma)(1+\alpha v)}{\alpha^2(\alpha+\beta)^4}\right] det\left(a_{i\bar{j}}\right)$$

Where:

$$N = \frac{(\alpha+\beta)^3}{4}, \ M = \frac{N}{\alpha} - \frac{1}{2\alpha^2+\gamma},$$

$$P = \frac{24\alpha(2\alpha^2+\gamma)}{(\alpha+\beta)^7(2\alpha^2+\gamma)+6(\alpha+\beta)^3(2\alpha^2+\gamma)(1+\alpha v)-24\alpha|\mu|^2}$$

Once obtained the metric tensor we must give the expressions of Chern-Finsler(c.n.c.).After some trivial calculus we have:

Proposition 3.3: Let (M,F) be a R-complex Hermitian space with *with (α,β)-metric L(α,β)=* $\frac{(\alpha+\beta)^4}{8}$. Then we have the following expressions of Chern-Finsler(c.n.c.):

$$N_j^i = N^{\alpha}{}_j^i - \frac{3}{2(\alpha+\beta)}\left[\frac{(2\alpha-\beta)(S-1)}{\alpha^2} + 3(\bar{\varepsilon} + +MNP\omega)\right]\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m\eta^i - \left(A\eta^i + Tb^i\right)\eta^l - -[(S+A)\eta^i + (T+\bar{M}NP)b^i]\bar{\eta}^m - \frac{1}{\alpha}\left(T - \frac{\bar{M}NP\bar{\gamma}}{2\alpha^2}\right)\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m b^i + PN\left(\frac{1}{\alpha}\frac{\partial l_{\bar{m}}}{\partial z^j} + \frac{\partial b_{\bar{m}}}{\partial z^j}\right)\cdot\left(Nb^i + M\eta^i\right)b^{\bar{m}}$$

Where:

$$S = \frac{1}{2\alpha^2+\gamma} + P|M|^2 S = \frac{1}{2\alpha^2+\gamma} + P|M|^2,$$

$$A = \frac{3}{2(\alpha+\beta)}\left(\frac{\gamma S}{\alpha} + \varepsilon S - 1 + MNP\omega\right)$$

$$T = \frac{3}{2(\alpha+\beta)}\left(N^2 P\omega + \frac{\bar{M}NP\bar{\gamma}}{\alpha} + \bar{M}NP\bar{\varepsilon} - 1\right)$$

As in [3] we have an example:

Example 2:

We construct this example like in [3] on $M = C^3$ where we set the metric

$$\alpha^2 = e^{z^1+\bar{z}^1}|\eta^1|^2 + e^{z^2+\bar{z}^2}|\eta^2|^2 + e^{z^1+\bar{z}^1 z+^2+\bar{z}^2}|\eta^3|^2$$

,the (1,0)-differential form $\varepsilon = e^{z^2}\eta^2$ and we have :

$$F = \frac{e^{z^1+\bar{z}^1}|\eta^1|^2 + e^{z^2+\bar{z}^2}|\eta^2|^2 + e^{z^1+\bar{z}^1 z+^2+\bar{z}^2}|\eta^3|^2}{\frac{1}{2}\left(e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2\right)}$$

4. R -COMPLEX FINSLER SPACE WITH QUADRATIC METRIC

In this case we consider $L(\alpha,\beta) = F^2 = \frac{(\alpha+\beta)^4}{\alpha^2}$

We compute the invariants and we have:

Proposition 4.1:*The invariants of this class of* R *-complex Finsler spaces with quadratic metric are:*

$$\rho_0 = 1 - \frac{\beta^4}{\alpha^4} + \frac{2\beta}{\alpha} - \frac{2\beta^3}{\alpha^3}, \rho_{-1} = -\frac{2\beta^3}{\alpha^5} - \frac{3\beta^2}{\alpha^3}$$

$$\rho_1 = 6\beta + \frac{2\beta^3}{\alpha^2} + 2\alpha + \frac{6\beta^2}{\alpha}, \mu_0 = 3 + \frac{3\beta^2}{\alpha^2} + \frac{6\beta}{\alpha}$$

$$\rho_{-2} = \frac{2\beta^4}{\alpha^6} + \frac{3\beta^3}{\alpha^5} - \frac{\beta}{\alpha^3}$$

Following the same steps like before we can compute:

Proposition 4.2:*The metric tensor field of an* R *-complex Hermitian Finsler space with* (α,β)*-metric* $L(\alpha,\beta)=\frac{(\alpha+\beta)^4}{\alpha^2}$ *is given by:*

$$g_{ij} = \left(1 - \frac{\beta^4}{\alpha^4} + \frac{2\beta}{\alpha} - \frac{2\beta^3}{\alpha^3}\right) a_{ij} + \left(\frac{2\beta^4}{\alpha^6} + \frac{3\beta^3}{\alpha^5} - \frac{\beta}{\alpha^3}\right) l_i l_j + \left(3 + \frac{3\beta^2}{\alpha^2} + \frac{6\beta}{\alpha}\right) b_i b_j + \left(-\frac{2\beta^3}{\alpha^5} + \frac{1}{\alpha} - \frac{3\beta^2}{\alpha^3}\right)\left(l_i b_j + b_i l_j\right)$$

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