THE APPROXIMATION OF A CONTINUOUS FUNCTION USING BERNSTEIN POLYNOMIALS

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Abstract: *The purpose of this article is to prove the Weierstrass theorem that relates to the limit of a convergent uniform polynomial array, in an (a,b) interval, using Bernstein polynomials. The first part of the paper briefly mentions notions connected to the best approximation of an* f *function given by the* P_n *polynomials. It is proven and concluded somehow geometrically, the form of the interpolating Bernstein polynomial* $B_n(x; f)$, *and with the help of the oscillation mode* $\omega(\delta)$ *of the* f *function, the superior limit of the difference* $|f(x) - B_n(x; f)|$ *is determined. The final part of the paper points out the best approximation given by the* $B_n(x; f)$ *polynomials for continuous functions, which close the Weierstrass theorem demonstration.*

Keywords: *approximation, polynomials, functions, continuity, boundedness*

1. INTRODUCTION

If a $f(x)$ function is given, we will say, by definition, that the distance $M(|f - P|)$ between this function and a $P(x)$ polynomial is the error or the approximation where $P(x)$ represents $f(x)$.

For all *n* degree polynomials, $M(|f - P|)$ has an $\mu_n(f)$ inferior margin, which represents the best approximation of the $f(x)$ function using *n* degree polynomials.

The problem of the best approximation is the following:

If a $f(x)$ is given, then the *n* degree polynomials are determined for which $M(|f - P|)$ that reaches its inferior margin $\mu_n(f)$ and this number is studied.

A $P(x)$ polynomial of *n* degree for which $\mu_n(f)$ is reached will be called the best approximation polynomial of *n* degree of the $f(x)$ function.

2. WEIERSTRASS THEOREM

Any continuous function on the (a,b) *interval is the limit of a uniformly convergent array of polynomials in this interval.*

From this theorem we could get $\lim_{n\to\infty+} \mu_n(f) = 0$ if the function *f* is continuous.

It is obvious that for any function *f* we have the following inequalities
$\mu_0 \geq \mu_1 \geq \ldots \geq \mu_n \geq \ldots$

So the limit

$$\lim_{n\to\infty+} \mu_n(f) = \mu$$

Exists and it is higher or equal to zero.

If $\mu = 0$, the polynomial array P_n converges absolutely and uniformly in the (a,b) interval. For a discontinuous function the result is $\mu \neq 0$.

Weierstrass' theorem states that for a continuous function we definitely have $\mu = 0$.

The important issue would be to prove the relation directly, based solely on the P_n polynomials properties.

Before proving the Weierstrass theorem, we state Tonelli' theorem, where the P_n polynomial is noted T_n.

Tonelli's Theorem

If the polynomial array $T_0(x; f), T_1(x; f),....,T_n(x; f),...$ converges absolutely and uniformly towards a continuous function, this function coincides with $f(x)$.

We assume that the polynomial array

$$T_0(x; f), T_1(x; f),....,T_n(x; f),...$$

Converges uniformly towards a continuous function $F(x)$ and that we have $\mu > 0$, then:

$$M(|f - F|) \leq M(|f - T_n|) + M(|F - T_n|) \leq \mu_n + M(|F - T_n|)$$

We easily deduce that

$$M(|f - F|) \leq \mu_n$$

As $f - F$ is a continuous function, to determine a $\delta > 0$ in any $\leq \delta$ length interval, the oscillation of this function has to be smaller than μ.

On the other hand, we can find a number $n > \frac{b-a}{\delta}$ so that we have

$$M(|F - T_n|) < \varepsilon < \frac{\mu}{2}$$

We know that there are $n+2$ points for which $\pm\mu_n$ is alternatively reached and, from the way n was reached, $n > \frac{b-a}{\delta}$, the resultant is the existence, among $n+2$ points, at least 2 points x' and x'' so that

$$|x' - x''| < \delta,$$

$$f(x') - T_n(x') = \mu_n$$

$$f(x'') - T_n(x'') = -\mu_n$$

Where

$$f(x') - F(x') = (f(x') - T_n(x')) + (T_n(x') - F(x')) > \mu_n - \varepsilon \geq \mu - \varepsilon > +\frac{\mu}{2}$$

$$f(x'') - F(x'') = (f(x'') - T_n(x'')) + (T_n(x'') - F(x'')) > -\mu_n + \varepsilon \geq -\mu + \varepsilon > -\frac{\mu}{2}$$

The result is that the oscillation of the $f-F$ function in the $(x';x'')$ interval is higher than μ, which is impossible. The hypothesis $\mu > 0$ is wrong. As a result, we must have $\mu = 0$ and then F coincides with f.

3. BERNSTEIN POLYNOMIALS

The purpose is to demonstrate the Weierstrass theorem using Bernstein polynomials.

The definition of Bernstein polynomials

We consider the interval (a,b), with $a<b$, $a,b \in \mathbf{R}$ which we divide in n equal parts and let

$$x_i = a + i \cdot \frac{b-a}{n}, \quad i = 0,1,2,\ldots,n$$

Where $x_0 = a$, $x_n = b$.

The definition of the interpolation polynomial

An n degree polynomial whose coefficients depend in a linear and homogenous way on the *(n+1)* values $f(x_i)$, $i = 0,1,2,\ldots,n$ is called an interpolation polynomial of n degree of the $f(x)$ function.

The purpose is to particularly study the Bernstein interpolation polynomial:

$$B_n(x;f) = \frac{1}{(b-a)^n}\sum_{i=0}^{n} C_n^i f(x_i)(x-a)^i (b-x)^{n-i}$$

3.1. A geometrical determination of the Bernstein polynomials

It is interesting to see how these Bernstein polynomials can be obtained in a rather geometrical way.

Let $X_0, X_1, \ldots, X_n$ be the representative points of the $f(x)$ function for $x = x_0, x_1, \ldots, x_n$, where $x_0 = a$, $x_n = b$, that is the coordinate points $X_i(x_i, f(x_i))$, $i = 0,1,2,\ldots,n$.

Let's build the polygonal line $X_0 X_1 \ldots X_n$.

We consider on the sides $X_0X_1, X_1X_2, \ldots, X_{n-1}X_n$ of the polygonal line the points $X'_0, X'_1, \ldots, X'_{n-1}$ that divide the sides in the same direction and in the same proportion, so that we can write

$$X_0X'_0 = X_1X'_1 = X_2X'_2 = \ldots = X_{n-1}X'_{n-1} = \frac{k}{n} \cdot \frac{b-a}{n}$$

Where k is considered an integer, $0 \le k \le n$.

On the polygonal line $X'_0 X'_1 \ldots X'_{n-1}$ we mark the polygonal line $X''_0 X''_1 \ldots X''_{n-2}$ in the same way, keeping the direction and the proportion of side division, therefore obtaining:

$$X'_0 X''_0 = X'_1 X''_1 = X'_2 X''_2 = \ldots = X'_{n-2} X''_{n-2} = \frac{k}{n} \cdot \frac{b-a}{n}$$

To continue this procedure, we mark the polygonal lines consecutively

$X_0^{(k)} X_1^{(k)} \ldots X_{n-k}^{(k)}$, $k = 3,4,\ldots,n$

The last polygonal line is reduced to a point, that is $X_0^{(n)}$.

Therefore, we obtain the equality:

$$X_0 X'_0 = X'_0 X''_0 = \ldots = X_0^{(n-1)} X_0^{(n)} = \frac{k}{n} \cdot \frac{b-a}{n}$$

Thus the abscissa of $X_0^{(n)}$ point is

$$x_k = a + k \cdot \frac{b-a}{n}$$

We note the $X_0^{(n)}$ point with X_k^* to be able to point out the number k and to calculate X_k^*'s ordinate.

We notice that if $i = 0$, X_i point coincides with X_0, respectively with X_n.

We note, generally, with y_k the ordinate of X_k point, with $y_r^{(s)}$ the ordinate of $X_r^{(s)}$ and with y_k^* the ordinate of X_k^*.

We have

$$y_r^{(s)} = \frac{(n-k) \cdot y_r^{(s-1)} + k \cdot y_{r+1}^{(s-1)}}{n}, r = 0,1,\ldots,n-s \text{ and } s = 1,2,\ldots,n-1$$

$$y_r^* = \frac{(n-k) \cdot y_0^{(n-1)} + k \cdot y_1^{(n-1)}}{n}.$$

From the first relation we consecutively deduce that

$$y_r^{(1)} = \frac{(n-k) \cdot y_r + k \cdot y_{r+1}}{n}$$

$$y_r^{(2)} = \frac{(n-k) \cdot y_r^{(1)} + k \cdot y_{r+1}^{(1)}}{n} = \frac{(n-k)^2 + 2k(n-k) \cdot y_{r+1} + k^2 \cdot y_{r+2}}{n^2}$$

And generally

$$y_r^{(s)} = \frac{1}{n^s} \sum_{i=0}^{s} C_s^i \cdot k^i (n-k)^{s-i} y_{r+i}; \; r = 0,1,\ldots,n-s.$$

$$y_k^* = \frac{1}{n^n} \sum_{i=0}^{n} C_n^i \cdot k^i (n-k)^{n-i} y_i$$

Coming back to the $B_n(x; f)$ polynomial we observe that

$$B_n(a + k \cdot \frac{b-a}{n}; f) = \frac{1}{n^2} \sum_{i=0}^{n} C_n^i \cdot k^i (n-k)^{n-i} f(x_i)$$

Thus the Bernstein polynomial $B_n(x; f)$ is Lagrange's polynomial that takes the y_k^* values in x_k point.

3.2. *Determining a superior limit for* $\left|f(x)-B_n(x;f)\right|$

The definition of the oscillation mode $\omega(\delta)$ of a *f* function

Let *f* be a continuous function on the (a,b) interval with $a<b$, $a,b\in\mathbf{R}$.

The oscillation mode of the *f* function is a δ function that, by definition, is given by the relation:

$$\omega(\delta)\overset{def}{=}\max\left|f(x')-f(x'')\right|$$

where x' and x'' are two points of the (a,b) interval so that $\left|x'-x''\right|\le\delta$.

Observations

a) $\omega(\delta)$ is a definite function for $0<\delta\le b-a$, non-decreasing and non-negative;

b) We have the inequality: $\left|f(x')-f(x'')\right|\le\omega\left(\left|x'-x''\right|\right)$

<u>*Statement*</u>

The necessary and sufficient condition for *f* to be continuous is that $\omega(\delta)\to 0$ for $\delta\to 0$.

The following observations concerning the statement above are made:

i. For $\varepsilon>0$ there are two x' and x'' points in the (a,b) interval with $x'<x''$ so that $\left|x'-x''\right|\le\delta$ And $\omega(\delta)-\varepsilon<\left|f(x')-f(x'')\right|$

ii. If we divide the interval (x',x'') in *k* equal parts in the points $x'=x_0;x_1;\ldots;x_{k-1}$; $x_k=x''$ we get

$$f(x')-f(x'')=\sum_{i=1}^{k}\left(f(x_i)-f(x_{i+1})\right)$$

Where

$$\left|f(x')-f(x'')\right|\le k\cdot\omega\left(\frac{\delta}{k}\right)$$

So

$$\omega(\delta)<k\omega\left(\frac{\delta}{k}\right)+\varepsilon$$

Whatever ε, and *k* being a positive integer.

Placing $k\cdot\delta$ instead of δ we get

$$\omega(k\delta)<k\omega(\delta)+\varepsilon<(k+1)\omega(\delta)$$

Whatever the positive *k* number so that $\delta\le b-a$ and $k\delta\le b-a$.

Therefore we obtain for $\delta\le b-a$

$$\left|f(x')-f(x'')\right|<\left[\frac{\left|x'-x''\right|}{\delta}+1\right]\omega(\delta)$$

Thus, the necessary and sufficient condition for *f* to be continuous is that $\omega(\delta)\to 0$ for $\delta\to 0$.

To continue, we plan, with the help of the oscillation module $\omega(\delta)$ to determine the superior limit for $\left|f(x)-B_n(x;f)\right|$.

Let's notice that $B_n(x;1)=1$, thus resulting:

$$\left|f(x)-B_n(x;f)\right| = \left|\frac{1}{(b-a)^n}\cdot\sum_{i=0}^{n}C_n^i\big(f(x)-f(x_i)\big)\cdot(x-a)^i(b-x)^{n-i}\right| \le$$

$$\le \frac{1}{(b-a)^n}\cdot\sum_{i=0}^{n}C_n^i\omega\big(\left|x-x_i\right|\big)\cdot(x-a)^i(b-x)^{n-i} <$$

$$< \left\{\frac{1}{\delta}\cdot\frac{1}{(b-a)^n}\cdot\sum_{i=0}^{n}C_n^i\left|x-x_i\right|\cdot(x-a)^i(b-x)^{n-i}+1\right\}\cdot\omega(\delta)$$

If we consider:

$$\Psi(x)=\frac{1}{(b-a)^n}\cdot\sum_{i=0}^{n}C_n^i\left|x-x_i\right|\cdot(x-a)^i(b-x)^{n-i}$$

and

$N_n=\max\limits_{x\in(a,b)}\Psi(x)$

and

$\delta=2N_n$, we determine a superior limit for $\left|f(x)-B_n(x;f)\right|$ as:

$\left|f(x)-B_n(x;f)\right|<\frac{3}{2}\omega(2N_n)$,

for $\delta\le b-a$.

3.3. The approximation given by the $B_n(x;f)$ polynomial

We can calculate the approximation given by the $B_n(x;f)$ polynomials.

Let's process first the function $\Psi(x)$.

In the (x_j, x_{j+1}) interval, we have:

$$\Psi(x)=\frac{1}{(b-a)^n}\cdot\sum_{i=0}^{j}C_n^i\big(x-x_i\big)\cdot(x-a)^i(b-x)^{n-i}+$$

$$+\frac{1}{(b-a)^n}\cdot\sum_{i=j+1}^{n}C_n^i\big(x_i-x\big)\cdot(x-a)^i(b-x)^{n-i}=$$

$$=\frac{2}{(b-a)^n}\cdot\sum_{i=0}^{j}C_n^i\big(x-x_i\big)\cdot(x-a)^i(b-x)^{n-i}$$

Because it can be easily checked that:

$$\sum_{i=0}^{n}C_n^i\big(x_i-x\big)\cdot(x-a)^i(b-x)^{n-i}=0$$

By doing the calculus, we find that

$$\Psi(x)=\frac{2}{(b-a)^n}\cdot C_{n-1}^j(x-a)^{j+1}(b-x)^{n-j}$$

The maximum of the polynomial in the (x_j, x_{j+1}) interval is reached for

$$x^*=\frac{(j+1)b+(n-j)a}{n+1}$$

And it has as value

$$\Psi(x^*)=2(b-a)C_{n-1}^j\frac{(j+1)^{j+1}\cdot(n-j)^{n-j}}{(n+1)^{n+1}}=2(b-a)\cdot\lambda_j$$

Where $\lambda_j = C_{n-1}^{j} \dfrac{(j+1)^{j+1} \cdot (n-j)^{n-j}}{(n+1)^{n+1}}$.

The following observation is useful

As the function $\left(\dfrac{x+1}{x}\right)^{x+1}$ is decreasing for $x \geq 1$, thus we have:

$$\left(\frac{j+2}{j+1}\right)^{j+2} > \left(\frac{n-j}{n-j-1}\right)^{n-j}$$

for $n > \dfrac{j+1}{2}$ or $\lambda_{j+1} > \lambda_j$.

Hence the function $\Psi(x^*)$ reaches its maximum for $j = \dfrac{n}{2}$ or $j = \dfrac{n-1}{2}$ if n is odd or even.

Thus, we have

$$N_n = 2(b-a)C_{n-1}^{n/2} \frac{\left(\frac{n}{2}+1\right)^{\frac{n}{2}+1}\left(\frac{n}{2}\right)^{\frac{n}{2}}}{(n+1)^{n+1}} \text{ for } n \text{ even}$$

$$N_n = \frac{(b-a)}{2^n} C_{n-1}^{n-1/2} \text{ for } n \text{ odd.}$$

It is proven that

$$\sqrt{2n-1} \cdot N_{2n-1} > \sqrt{2n+1} \cdot N_{2n+1}$$

$$N_1 = \frac{(b-a)}{2}, N_3 = \frac{(b-a)}{4}$$

where

$$N_{2n+1} < \frac{b-a}{2\sqrt{2n+1}}$$

$$N_{2n+1} \leq \frac{\sqrt{3}(b-a)}{4\sqrt{2n+1}}$$

for $n \geq 1$.

For n even, we have

$$N_{2n} = N_{2n+1} \frac{(n+1)^{n+1} n^n}{(2n+1)^{2n+1}} 2^{2n+1} < N_{2n+1} \frac{2^{2n+1}(n+1)}{(2n+1)^{2n+1}} \left(\frac{2n+1}{2}\right)^{2n} =$$

$$= N_{2n+1} \frac{2(n+1)}{2n+1} \leq \frac{\sqrt{3}(b-a)}{4\sqrt{2n+1}} \cdot \frac{2(n+1)}{2n+1} = \frac{1}{2} \cdot \frac{\sqrt{3}(n+1)(a-b)}{(2n+1)\sqrt{2n+1}} < \frac{b-a}{2\sqrt{2n}}$$

So generally

$$N_n \leq \frac{b-a}{2\sqrt{n}}$$

The relation becomes

$$\left| f(x) - B_n(x; f) \right| < \frac{3}{2} \omega\left(\frac{b-a}{\sqrt{n}}\right)$$

If the function f is continuous $\omega\left(\frac{b-a}{\sqrt{n}}\right) \to 0$ for $n \to \infty$, Weierstrass theorem is demonstrated as well. Moreover, the best approximation of a continuous function is seen using n degree polynomials, that is μ_n, is at least of $\omega\left(\frac{b-a}{\sqrt{n}}\right)$ degree.

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