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ANALYSIS OF THE ROBUSTNESS OF THE AUTOMATIC CONTROL SYSTEMS

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Abstract: *Modern control systems act in real environment, and must be able to handle problems of reference signal tracking and disturbance attenuation simultaneously. This paper highlights theory of the analysis of the robustness of the automatic control systems used in automatic flight control systems.*

Keywords: *modeling uncertainties, loop shaping, robust stability, robust performance.*

1. INTRODUCTION

This paper is lean upon early work of the author dealing with analysis of the robustness of the automatic control systems in general [1], and on paper representing application of this theory to analyze stability augmentation system of the aircraft [2].

Control system is designed to work in real environment. The controller is often designed for the system with simplified mathematical model. Nonlinearities are often omitted or linearized and the controller is synthesized for the linearized system. However, controller works with the real nonlinear system. Sensor dynamics, actuator and motor dynamics are also simplified or neglected. Dynamics of the plant contains high frequency elastic oscillation modes, which can be neglected.

The control system, which is able to work in real environment, is called for *robust* one. It means that controller is able to meet design requirements not only for the simplified plant model used during synthesis but for its family representing all possible plant models including both nonlinearities and high frequency dynamics. In this case control system has robust stability and robust performance.

For instance, automatic flight control systems are designed to work also in extreme flight conditions, e. g. extremely high or low air temperature and pressure, load factors, maneuvers, turbulent air etc. Flight control system is able to meet all design requirements in any flight conditions.

Mathematical description of the deterministic systems is given in [5, 6, 7]. The stochastic dynamical systems and signals are analyzed in [3, 4, 5, 7, 8, 9, 10]. In [4, 8, 9, 10] there are many applications of robust control and modeling robust control systems. In [10] an example of robust controller synthesis for fighter aircraft is presented when high frequency dynamics of the aircraft fuselage is added to that of the rigid one. In [13] mathematical models including static and dynamical ones are given. Part II. of [13] deals with modeling of stochastical systems, and with design of the robust dynamic controller.

Chapter 3 gives more general interpretation of the mathematical models given for the SISO[1] control systems and derives matrix equations for MIMO[2] control systems.

Chapter 4 shows loop shaping problem, which is about bounding sensitivity transfer function, and closed loop system transfer function, and gives some remarks on this problem.

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[1] Single Input – Single Output
[2] Multi Input – Multi Putput

Chapter 5 is for defining mathematical models of the uncertainties playing active role during synthesis of the robust controller.

Chapter 6 is dealing with robust stability of the control systems, i. e. giving mathematical models both for the additive and multiplicative uncertainties. This section also deals with derivation stability margins, which are very important quantitative measures during control systems' analysis and design.

2. DYNAMIC PERFORMANCES OF THE SISO SYSTEMS

Block diagram of the SISO control system can be seen in Fig 1 [1].

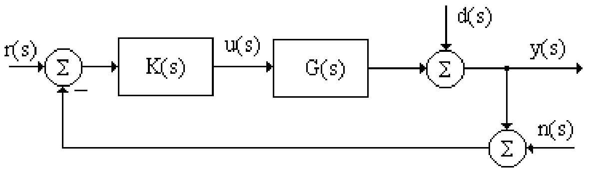

Fig 1. Block Diagram of the SISO control system.

In Fig 1: *r(s)* - reference signal, *d(s)* – external disturbance, *n(s)* – sensor noise, *G(s)* – transfer function of the plant, *K(s)* – transfer function of the controller, *u(s)* – input vector, *y(s)* – output signal. Using Fig 1 the output signal can be derived as:

$$y(s) = \frac{K(s)G(s)}{1+K(s)G(s)} r(s) + \frac{1}{1+K(s)G(s)} d(s) - \frac{K(s)G(s)}{1+K(s)G(s)} n(s) \quad (2.1)$$

Let us introduce the following substitutions: $L(s) = K(s)G(s)$ – open loop transfer function, $S(s) = \frac{1}{1+K(s)G(s)}$ – sensitivity transfer function, $T(s) = \frac{K(s)G(s)}{1+K(s)G(s)}$ – closed loop complementary transfer function (closed loop transfer function). From equations given above it is evident that

$$S(s) + T(s) = 1 \quad (2.2)$$

For achieving prescribed reference signal tracking ability sensitivity transfer function *S(s)* should has small value in given frequency domain, i. e. open loop transfer function is large. For achieving necessary noise suppressing ability sensitivity transfer function *S(s)* must have small value in the frequency domain, in which external disturbance *d(s)* acts [4, 8, 9].

Sensor noises are said to be well damped if the closed loop transfer function *T(s)* has small values in the given frequency domain, i. e. open loop transfer function is also has small value.

Bode diagrams of the sensitivity transfer function *S(s)* and the closed loop complementary transfer function *T(s)* can be seen in Fig 2.

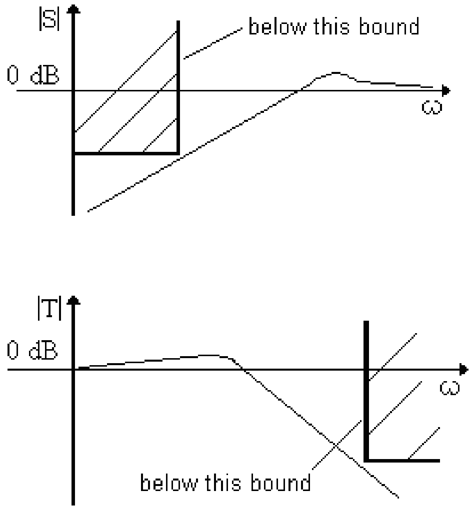

Fig 2. Bounds for $|S(j\omega)|$ and $|T(j\omega)|$

In low frequency domain $|S(j\omega)|$ must be kept small, in high frequency domain its absolute value goes to unity. In low frequency domain $|T(j\omega)|$ must be kept unit value, in high frequency domain is bounded for 'good' noise suppressing ability.

For the SISO control system these simultaneous requirements given above determine the shape of the open loop Bode diagram illustrated in Fig 3. In low frequency domain, in which reference signal and the disturbance act, open loop gain must be kept large. In high frequency domain open loop gain must be small for 'good' noise suppressing ability. For the control of the gain and phase margins at the crossover frequency slope of the Bode plot is – 20 dB/decade.

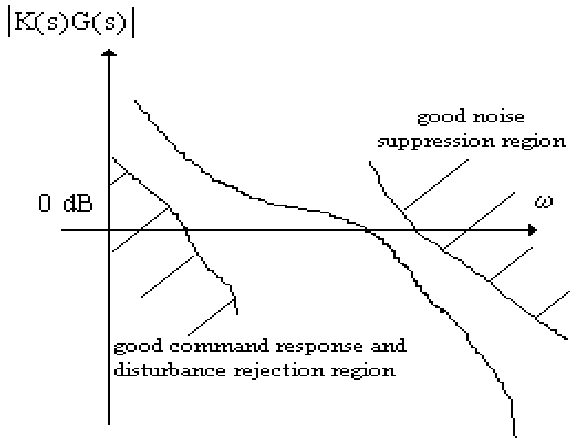

Fig 3. Desirable shape of the open loop system Bode diagram.

Summing up what has been said before: in the SISO control system nominal performances set limits on the size of the open loop gain $L(s) = K(s)G(s)$ [1, 4, 8, 9].

3. DYNAMIC PERFORMANCES OF THE MIMO SYSTEMS

Most of the control systems are MIMO ones and the state space method should be applied for its analysis and design. In this case all input signals are vectors. In the MIMO control system we deal with so-called transfer function matrices. For the evaluation of the size of matrices there is widely applied the *matrix singular value* method. For the MIMO control system eq. (2.1) may be rewritten in following manner [1, 4, 6, 8]:

$$\mathbf{y}(s) = \frac{\mathbf{G}(s)\mathbf{K}(s)}{[\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)]}\mathbf{r}(s) - \frac{\mathbf{G}(s)\mathbf{K}(s)}{[\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)]}\mathbf{n}(s) + \frac{1}{[\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)]}\mathbf{d}(s) \quad (3.1)$$

The sensitivity and the closed loop sensitivity transfer function matrices can be determined as follows:

$$\mathbf{T}(s) = \frac{\mathbf{G}(s)\mathbf{K}(s)}{[\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)]}, \qquad \mathbf{S}(s) = \frac{1}{[\mathbf{I}+\mathbf{G}(s)\mathbf{K}(s)]} \quad (3.2)$$

Nominal performance criterions for the SISO and the MIMO control systems are summarized in Table 1. Subscript 'm' denotes the largest singular values of the matrices [1].

Dynamic Performances of the SISO and MIMO systems Table 1.

	Low Frequency Domain		High Frequency Domain	
	SISO	**MIMO**	**SISO**	**MIMO**
Reference Signal Tracking	$\lvert K(s)G(s)\rvert \gg 1$ or $\lvert S(s)\rvert \ll 1$	$\sigma(\mathbf{K}(s)\mathbf{G}(s)) \gg 1$ or $\sigma_m(\mathbf{S}(s)) \ll 1$		
Disturbance Rejection	$\lvert K(s)G(s)\rvert \gg 1$ or $\lvert S(s)\rvert \ll 1$	$\sigma(\mathbf{K}(s)\mathbf{G}(s)) \gg 1$ or $\sigma_m(\mathbf{S}(s)) \ll 1$		
Noise Suppression			$\lvert K(s)G(s)\rvert \ll 1$ or $\lvert T(s)\rvert \ll 1$	$\sigma_m(\mathbf{K}(s)\mathbf{G}(s)) \ll 1$ or $\sigma_m(\mathbf{T}(s)) \ll 1$

The transfer function matrices are functions of the complex frequency *s*, their singular values are frequency dependent ones: singular values determined for $s = j\omega$ can be plotted versus frequency. Singular value frequency plots are generalizations of Bode magnitude plots for the MIMO systems.

4. LOOP SHAPING OF THE CONTROL SYSTEMS

It is known from control theory that dynamic performances of the feedback control systems can be translated into specifications on the sensitivity transfer function *S(s)* and the closed loop transfer function *T(s)*. The control system design methodology based upon determination of the appropriate bounds on *S* and *T* called *Loop Shaping*. This procedure can be applied for multivariable systems when we shape singular values of matrices ***S(s)*** and ***T(s)***. Let us consider the feedback system block diagram represented in Fig 4 for the formulation of problem of the loop shaping [1, 4, 7, 8, 9].

Firstly, let us consider *d(s)* and *n(s)* for the inputs. Using Fig 4 – for the SISO control system – following equations will take place:

$$\mathbf{Z}_1(s) = \mathbf{W}_S[1 + \mathbf{K}(s)\mathbf{G}(s)]^{-1} d(s)$$

$$\text{or } \mathbf{Z}_1(s) = \mathbf{W}_S \mathbf{S}(s) d(s) \qquad (4.1)$$

$$\mathbf{Z}_2(s) = \mathbf{W}_T[1 + \mathbf{K}(s)\mathbf{G}(s)]^{-1} \mathbf{K}(s)\mathbf{G}(s) n(s)$$

$$\text{or } \mathbf{Z}_2(s) = \mathbf{W}_T \mathbf{T}(s) n(s)$$

Secondly, let us consider for the input reference signal of *u(s)*. It yields to the following formula:

$$\mathbf{Z}_2(s) = \mathbf{W}_T \mathbf{T}(s)\, u(s) \qquad (4.\ 2)$$

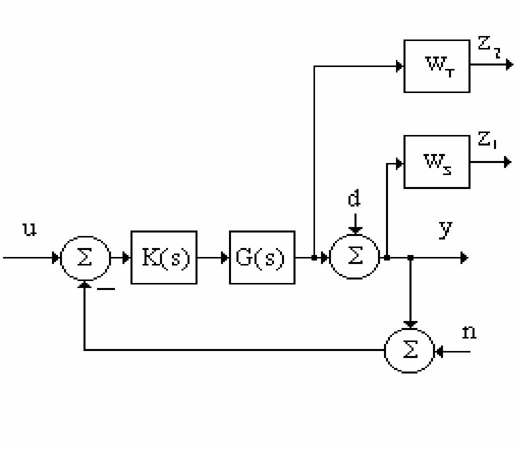

Fig 4. Loop shaping of the feedback control system.

In eqs. (4.1) and (4.2) $\mathbf{W}_S$ and $\mathbf{W}_T$ are weighting matrices that are used to bound **S** and **T**. Typical shapes for the ***S(s)***, ***T(s)***, $\mathbf{W}_S(s)$ and $\mathbf{W}_T(s)$ are given in Fig 5 [1].

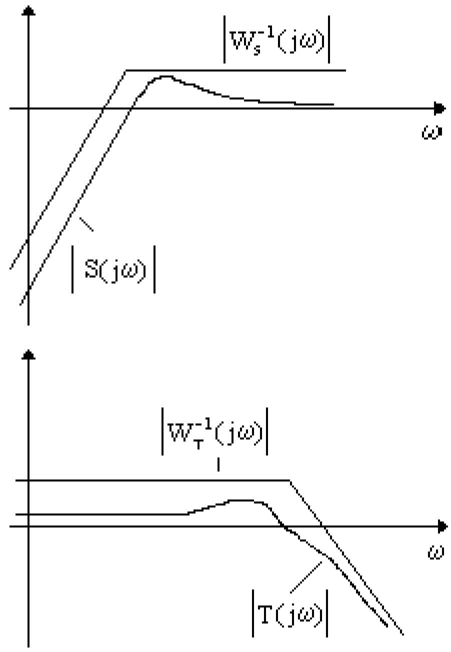

Fig 5. Shapes for **S** and **T** and their weights.

From Fig 5 following equations can be derived

$$|\,\mathbf{S}(s)\,| \le \left|\mathbf{W}_S^{-1}\right|, \text{ or } \left|\mathbf{W}_S\ \mathbf{S}(s)\right| \le 1 \qquad (4.3)$$

In control theory this inequality is known as the *weighted sensitivity problem*.

Secondly, for the closed loop transfer function **T**(s) takes place the following inequality:

$$|\,\mathbf{T}(s)\,| \le \left|\mathbf{W}_T^{-1}\right|, \text{ or } \left|\mathbf{W}_T\ \mathbf{T}(s)\right| \le 1 \qquad (4.4)$$

This problem is known in modern control theory as the *weighted complementary sensitivity problem*.

The simultaneous application and satisfaction of both constraints is called *mixed sensitivity problem* [1, 2, 5, 6, 7, 8, 9].

5. MODELLING OF UNCERTAINTIES IN CONTOL THEORY

Model uncertainty can be divided into two categories: structured and unstructured uncertainties. Structured uncertainty is the modeled one and has ranges and bounds on it. Unstructured uncertainty is the less-known one. We can assume that its frequency response lies between two bounds [1, 2, 5, 6, 7, 8, 9, 10, 11].

Unstructured uncertainty can be modeled in two different ways. One can discuss additive or multiplicative uncertainties. Let the nominal system model is denoted by $G(s)$. The actual true system is defined by $\tilde{G}(s)$. The actual system can be modeled as sum of nominal system plus additive uncertainty model:

$$\tilde{G}(s) = G(s) + \Delta_a(s) \qquad (5.1)$$

From eq. (5.1) the model of the additive uncertainty can be derived as:

$$\Delta_a(s) = \tilde{G}(s) - G(s) \qquad (5.2)$$

Additive uncertainty can be represented using eq. (5.1) and it can be seen in Fig 6.

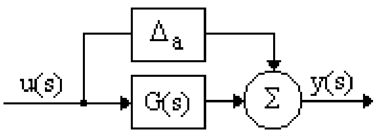

Fig 6. Block diagram of the additive uncertainty model.

Additive uncertainty model is often used in automatic flight control system to model aeroelastic high frequency dynamics of the aircraft fuselage [2, 5, 12, 13]. Additive uncertainty represents absolute error in the model e.g. omitted high frequency elastic motion dynamics.

In multiplicative uncertainty case one can find the true model of the system as:

$$\tilde{G}(s) = \left(1 + \Delta_m(s)\right) G(s) \qquad (5.3)$$

Multiplicative uncertainty can be built using eq. (5.3). It can be represented at the plant input or at the plant output. Block diagram of multiplicative uncertainty can be seen in Fig 7.

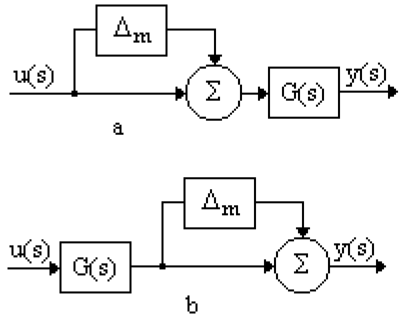

Fig 7. Block diagram of the multiplicative uncertainty model.
'a' – uncertainty at the plant input, 'b' – uncertainty at the plant output

Multiplicative uncertainty represents relative error in the model and it is used more often than additive one.

6. ROBUST STABILITY OF CONTROL SYSTEMS

Let us consider a feedback control system containing a plant and the compensator designed for the nominal plant $G(s)$. The compensator is *robustly* stabilizes the system if the closed loop control system remains stable for the true plant $\tilde{G}(s)$.

Robust stability conditions can be derived from variation of the Nyquist stability criterion or from the so-called *small gain theorem*. This theory states that, for the closed loop stability the open loop gain $|G(s)K(s)|$ is small.

The small-gain theorem guarantees internal stability. It means that all possible closed loop transfer functions are stable and all internal signals are bounded for bounded inputs.

From Chapter 2 it is known that for good command performance and for good disturbance rejection in the low frequency domain the open loop gain must be larger than one (see Figure 3). Hence, the control system satisfying this theorem will have poor dynamic performances. Inspite of this it is possible to apply the small-gain theorem for control systems with additive and multiplicative uncertainties.

The small-gain theorem is mainly used for answering the following two questions. The first is, if the given uncertainty is stable and bounded will the closed loop system be stable for this uncertainty? The second one is, for the given control system what is the smallest uncertainty destabilizing the closed loop control system?

Consider a system with nominal plant $G(s)$ and the compensator (see Fig 8). The plant and the compensator are supposed to be stable ones.

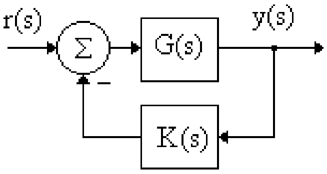

Fig 8. Block diagram of the feedback control system.

Using Nyquist stability criterion the closed loop control system is stable if and only if takes place the following inequality:

$$|G(s)K(s)| < 1 \tag{6.1}$$

Left side of the inequality can be rewritten as:

$$|G(s)K(s)| \leq |G(s)||K(s)| \tag{6.2}$$

The closed loop stability condition can be derived from eqs. (6.1) and (6.2). We have for this criterion:

$$|G(s)||K(s)| < 1 \tag{6.3}$$

Let us use the small-gain theorem for derivation of conditions of robust stability of control system under multiplicative uncertainty at the plant output. Consider the feedback system shown in Fig 9a. To derive the block diagram of the feedback system in Fig 8 it is necessary to determine the transfer function seen by the uncertainty. For these refer to Fig 9b and the transfer function *M(s)* (see Fig 9c) between 'input' and 'output' is given by [1, 5]:

$$M(s) = \frac{-G(s)K(s)}{1+G(s)K(s)} \tag{6.4}$$

The small-gain theorem states that if the above transfer function and the uncertainty transfer function are stable the closed loop control system will be robustly stable if and only if [1, 4, 5]:

$$|\Delta_m(s)| < \frac{1}{\left|G(s)K(s)[1+G(s)K(s)]^{-1}\right|}, \tag{6.5}$$

or in other representation

$$|\Delta_m(s)| < \frac{1}{|T(s)|}. \tag{6.6}$$

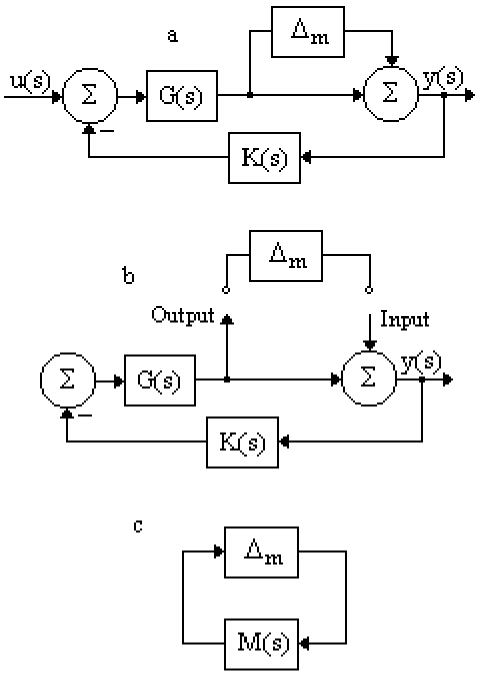

Fig 9. Feedback control system with multiplicative uncertainty.

Eqs. (6.5) and (6.6) can be used to answer the first of two questions posed earlier. If the uncertainty is bounded by the given scalar γ, one can have the following inequality:

$$|T(s)| < \frac{1}{\gamma}, \text{ or } |\gamma\, T(s)| < 1 \tag{6.7}$$

The second question of two posed before is about finding the smallest stable multiplicative uncertainty, which will destabilize the closed loop system. It is known that uncertainty must be smaller than $\frac{1}{|T(s)|}$, i. e. it must be smaller than minimum of $\frac{1}{|T(s)|}$. For the minimum of the right side of eq. (6.6) we must maximize $T(s)$. The maximum of $T(s)$ over all possible frequencies is its peak value. The smallest uncertainty destabilizing the feedback system is given by [6] to be:

$$MSM = \frac{1}{M_r},$$

where $M_r = \sup_{\omega} | T(j\omega)|$. (6.8)

In eq. (6.8) *MSM* denotes the *Multiplicative Stability Margin*.

The *supremum* of $| T(j\omega)|$ is equal to the maximum of the function being investigated on the given frequency range.

For the MIMO feedback system the size of the smallest destabilizing multiplicative uncertainty can be derived as follows:

$$\bar{\sigma}[\Delta_m(j\omega)] = \frac{1}{\bar{\sigma}[T(j\omega)]}. \qquad (6.9)$$

Using the same approach conditions of robust stability under additive uncertainty can be determined. In this particular case transfer function seen by the uncertainty is given as follows

$$M(s) = \frac{-K(s)}{1+G(s)K(s)} \qquad (6.10)$$

The feedback system will be robustly stable if takes place the following inequality:

$$|\Delta_a(s)| < \frac{1}{\left|K(s)[1+G(s)K(s)]^{-1}\right|}, \qquad (6.11)$$

or, in other manner

$$|\Delta_a(s)| < \frac{1}{|K(s)S(s)|}. \qquad (6.12)$$

If the additive uncertainty is stable and bounded by

$$|\Delta_a(s)| < \frac{1}{\gamma}. \qquad (6.13)$$

The closed loop robust stability can be guaranteed if

$$|K(s)S(s)| < \frac{1}{\gamma}, \text{ or } |\gamma\, K(s)S(s)| < 1 \qquad (6.14)$$

The *Additive Stability Margin (ASM)* can be defined by [6] as follows:

$$ASM = \frac{1}{\sup_{\omega}|K(j\omega)S(j\omega)|}. \qquad (6.15)$$

For the MIMO feedback system the size of the smallest additive uncertainty destabilizing the feedback system can be derived as follows:

$$\bar{\sigma}[\Delta_a(j\omega)] = \frac{1}{\bar{\sigma}[K(j\omega)S(j\omega)]}. \qquad (6.16)$$

It is easily can be seen that for protection against destabilizing multiplicative uncertainties MSM must be large, the complementary sensitivity must be small. It leads to good noise suppression but conflict with reference signal tracking and disturbance rejection. The transfer function of ASM is that of determining control energy.

7. SUMMARY

The paper dealt with dynamic performances of the SISO and MIMO feedback systems and with its main equations. The sensitivity and closed loop sensitivity transfer functions have been involved to determine the desirable shape of the open loop control system Bode diagram. Shapes of these functions were determined so as to meet dynamic performances of the feedback system.

Two kinds of uncertainties were derived for determination if controller is able robustly stabilize the true plant with given uncertainty. For derivation of smallest uncertainty destabilizing the closed loop control system the multiplicative and additive stability margins also were determined.

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