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MAX-STABLE DISTRIBUTION – part II

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Abstract: *In this part we continue the presentation from part I . The present paper demonstrates that this class of "max-stable'' distributions is made up of distributions with extreme values and each max-stable distributions matches one of the parametric forms corresponding to the distributions known as Gumbelle, Frechet, Weibull. In the end we present an important result for* $(\varepsilon_n)_n$ *independent random variables,identically and normally distributed, the series of random variables* $(M_n)_n$ *weakly converges to a Gumbell allotment.*

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1. THEOREM OF EXTREME TYPES

Theorem 1.2

May $M_n = \max(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ where ε_i are independent and identically distributed random variables. If $a_n > 0$ and b_n constants and

$$P(a_n(M_n - b_n) \le x) \xrightarrow{w} G(x) \qquad (1.1)$$

For some nondecreasing G functions, then G coincides with one of the three types of extreme values previously defined.

Reciprocally, each G distribution function of the extreme value type which appears as a limit of type (1.1) is unique distribution function for function ε_i.

Proof

If (1.1) is valid, Theorem 3.1 shows that G is max-stable and consequently from theorem 1.1 is of extreme value type. Reversibly, if G is of extreme value type, is max-stable from theorem 1.1 and theorem 3.1(b) shows that $G \in D(G)$.

If $\varepsilon_1, \varepsilon_2, ...$ are not necessarily independent, but $M_n = \max(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ has an asymptotical distribution G in the bearing of (1.4), then (3.1) is true for $k = 1$, where F_n is distribution function of M_n. If one can show that if (3.1) is true for $k = 1$, then it is true for all *k*, so it will result that G is max-stable from theorem 3.1 (a) and as a result G is extreme value type.

Thus our focus when considering dependent cases will consist only in showing that under the correct assumptions, the truth from (3.1) for $k = 1$ implies the truth from (3.1). For all *k*, from where, again, it results the Theorem of extreme types.

Coming back to the case independent and identically distributed random variables we note that theorem 1.1 assumes that $a_n(M_n - b_n)$ has a nondecreasing limit distribution function G and than it demonstrates that G must have one of the three presented forms. It is easy to build the sequences $\{\varepsilon_n\}$, independent and identically distributed random variables for which there is not such a G. in order to see an easy example, for this case it is convenient to use the notation $x_F = \sup\{x; F(x) < 1\}(<\infty)$.

That means that $F(x) < 1$ for all $x < x_F$ and $F(x) = 1$ for all $x \ge x_F$. We assume that each ε_n has a distribution function which is such as $x_F < \infty$ and thus F has at x_F a continuity point i.e. $F(x_{F-}) < 1 = F(x_F)$. Then it results that if $\{u_n\}$ is any sequence and $P\{M_n \le u_n\} \to \rho$, then $\rho = 0$ or 1. Thus if $P\{a_n(M_n - b_n) \le x\} \to G(x)$, it follows taking a $u_n = \frac{x}{a_n} + b_n$, that $G(x) = 0$ or 1 for each *x*, so that *G* is nondecreasing.

2. CONVERGENCE OF $P\{M_n \le u_n\}$

We have taken into consideration convergence of the probabilities of the form $P\{a_n(M_n - b_n) \le x\}$ which can be rewritten as $P\{M_n \le u_n\}$, where $u_n = u_n(x) = \frac{x}{a_n} + b_n$.

The convergence was asked for all *x*. On the other hand, we are interested in considering the sequences $\{u_n\}$ which can be non dependent on any parameter *x* or can be functions more complicated than the linear one considered above.

The next theorem is almost trivial in the context independent and identically distribution but it is also very important and will be extended through important means in order to be applied (stationary) to the dependent sequences and continuous time processes.

Theorem 2.1

May $\{\varepsilon_n\}$ a selection of independent and identically distributed random variables. May $0 \le \tau \le +\infty$ with the assumption that if $\{u_n\}$ is a sequence of real numbers for which:

$$\lim_{n\to\infty} n(1 - F(u_n)) = \tau \qquad (2.1)$$

Then

$$\lim_{n\to\infty} P\{M_n \le u_n\} = \mathrm{e}^{-\tau} \qquad (2.2)$$

Conversely, if (2.2) holds true for a τ with $0 \le \tau \le +\infty$ then the relation (2.1) it is true.

Proof

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If $0 \le \tau \le +\infty$ so that:

$$P\{M_n \le u_n\} = F^n(u_n) = \{1 - (1 - F(u_n))\}^n \quad (2.3)$$

According to the hypothesis, from $\lim_{n\to\infty} n(1 - F(u_n)) = \tau \Rightarrow$ there is n_r so that any $n \ge n_r$

$$|n(1 - F(u_n)) - \tau| < r$$

$$-r < n(1 - F(u_n)) - \tau < r \,|+\tau$$

$$\tau - r < n(1 - F(u_n)) < r + \tau \,|:n$$

$$\frac{\tau - r}{n} < 1 - F(u_n) < \frac{r+\tau}{n} \,\Big|(-1)$$

$$-1 + \frac{\tau - r}{n} < -F(u_n) < -1 + \frac{r+\tau}{n} \,\Big|(-1)$$

$$1 - \frac{r+\tau}{n} < F(u_n) < 1 - \frac{r-\tau}{n}$$

$$1 - \frac{\tau}{n} - \frac{r}{n} < F(u_n) < 1 - \frac{\tau}{n} + \frac{r}{n}$$

Therefore

$$P(M_n \le u_n) = \left(1 - \frac{\tau}{n} + 0\left(\frac{1}{n}\right)\right)^n \Rightarrow$$

$$\lim_{n\to\infty} P\{M_n \le u_n\} = 1^\infty = \mathrm{e}^{\lim\limits_{n\to\infty}\left(-\frac{\tau}{n} + 0\left(\frac{1}{n}\right)\right)\cdot n}$$

$$= \mathrm{e}^{\lim\limits_{n\to\infty}\left(-\tau + n\cdot 0\left(\frac{1}{n}\right)\right)} = \mathrm{e}^{-\tau}.$$

"<="

From

$$\lim_{n\to\infty} P(M_n \le u_n) = \lim_{n\to\infty}\{1 - (1 - F(u_n))\}^n =$$

$$= e^{-\tau} \Rightarrow (1 - F(u_n)) = 0$$

For $1 - F(u_{nk})$ limited by *O* for the sequence $\{n_k\}$ according to the relation (2.3) from which it results:

$$\lim_{n\to\infty} P(M_n \le u_n) = 0$$

$$\left.\begin{aligned} \ln P(M_n \le u_n) &= \ln \mathrm{e}^{-\tau} = -\tau \\ \ln P(M_n \le u_n) &= n \ln(1 - (1 - F(u_n))) \end{aligned}\right\} \Rightarrow$$

$$n \ln(1 - (1 - F(u_n))) \to -\tau$$

$$n(1 - F(u_n))(1 + O(1)) \to \tau$$

Finally if $\tau=\infty$ and (2.1) is true but (2.2) is not true, there must be a sequence $\{n_k\}$ so that $P\{M_n \le u_n\} \to e^{-\tau}$ while $k \to \infty$ for $\tau' < \infty$. But the relation (2.2) implies (2.1) with n_k replacing n so that $n_k(1-F(u_{nk})) \to \tau' < \infty$, contradicting the assumption that (2.2) is true for $\tau=\infty$. Similarly (2.2) implies (2.1) when $\tau=\infty$.

Corollary 2.2

(1) $M_n \to x_F (\le +\infty)$ with probability 1 for $n \to \infty$.

(2) If $x_F < \infty$ and $F(x_{F-}) < 1$ and if for the sequence $\{u_n\}$, $P(M_n \le u_n) \to \rho$ then $\rho = 0$ or $\rho = 1$.

Proof

If $\lambda < x_F (\pm\infty)$, $1 - F(\lambda) > 0$ so that (2.1) is true for $u_n = \lambda, \tau = \infty$ and from (2.2) we obtain $\lim\limits_{n\to\infty} P(M_n \le \lambda) = 0$. But $P(M_n > x_F) = 0$ for any n, from where it results $M_n \to x_F$ in probability. As $\{M_n\}$ is monotonous and convergent it results that $M_n \to x_F$ and point (1) is proved.

Assuming that $x_F < \infty$ and $F(x_{F-}) < 1$. Let the sequence $\{u_n\}$ so that $P(M_n \le u_n) \to \rho$. As $\rho \in [0,1]$ we can write $\rho = e^{-\tau}, 0 \le \tau < \infty$ and from the theorem 2.2 we obtain $n(1-F(u_n)) \to \tau$.

If $u_n < X_F$ for an infinite number of values of n and because $1-F(u_n) \ge 1 - F(x_{F-}) > 0$ we have $\tau = \infty$ and $u_n \ge X_F$ and we obtain

$$\left.\begin{array}{l} n(1-F(u_n)) = 0 \\ n(1-F(u_n)) \to \tau \end{array}\right\} \text{ results that } \tau = 0 .$$

Therefore $\tau = \infty$ or $\tau = 0$ and consequently $\rho = 0$ or $\rho = 1$ Q.E.D.

We go on bringing into discussion the interest domain of distributions with extreme values. The normal selections are important and consequently it is demonstrated that theorem 2.1 can be used directly to obtain asimpthotic laws of type Tip I for normal independent and identically distributed selections.

We consider J the normal standard distributive function and Φ the density function corresponding to the mention that there will be repetitively used the known relation of connection:

$$1 - J(u) \approx \frac{\Phi(u)}{u} \text{ when } u \to \infty . \qquad (2.4)$$

Theorem 2.3

If $\{\varepsilon_n\}$ is a normal selection independent and identically distributed (standard) of random variables, then the asimpthotical distribution of $M_n = \max(\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n)$ is of Type I and

$$P\{a_n(M_n - b_n) \le x\} \to \exp\left(-e^{-x}\right) \qquad (2.5)$$

Where $a_n = (2\ln n)^{1/2}$ and

$$b_n = (2\ln n)^{1/2} - \frac{1}{2}(2\ln n)^{-1/2}(\ln\ln n + \ln 4\pi)$$

Proof

We choose $\tau = e^{-x}$ in relation (2.1), then

$$\left.\begin{array}{l} 1 - J(u_n) = \dfrac{1}{n} e^{-x} \\ 1 - J(u_n) \approx \dfrac{\Phi(u_n)}{u_n} \end{array}\right| \Rightarrow \frac{\frac{1}{n} e^{-x} \cdot u_n}{\Phi(u_n)} \to 1 \text{ and by}$$

looking up its logarithm we obtain:

$$\left.\begin{aligned}-\ln n-x+\ln u_n-\ln\Phi(u_n)\to 0\\ \Phi(u_n)=\frac{1}{\sqrt{2\pi}}e^{-\frac{u_n^2}{2}}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow -\ln n+\ln u_n+\frac{1}{2}\ln 2\pi+\frac{u_n^2}{2}\to 0 \quad (2.6)$$

As $\frac{u_n^2}{2\ln n}\to 1$ we obtain

$2\ln u_n-\ln 2-\ln(\ln u)\to 0\Rightarrow$

$\ln u_n=\frac{1}{2}(\ln 2+\ln(\ln u))+0(1)$ (*)

Using (*) in (2.6) we obtain

$$-\ln n-x+\frac{1}{2}(\ln 2+\ln(\ln u))+0(1)+\frac{1}{2}\ln 2\pi+\frac{u_n^2}{2}\to 0$$

$$\frac{u_n^2}{2}=x+\ln n-\frac{1}{2}\ln 4\pi-\frac{1}{2}\ln(\ln n)+0(1)$$

$$u_n^2=2x+2\ln n-\ln 4\pi-\ln(\ln n)+\theta(2)$$

$$u_n^2=(2\ln n)\left(\frac{x}{\ln n}+1-\frac{\ln 4\pi-\ln(\ln n)}{2\ln n}+\theta\left(\frac{1}{\ln n}\right)\right)$$

$$u_n(2\ln n)^{\frac{1}{2}}\left(1+\frac{x-\frac{1}{2}\ln 4\pi-\frac{1}{2}\ln(\ln n)}{\ln n}+\theta\left(\frac{1}{\ln n}\right)\right)^{\frac{1}{2}}$$

$$u_n=(2\ln n)^{\frac{1}{2}}\left(1+\frac{x-\frac{1}{2}\ln 4\pi-\frac{1}{2}\ln(\ln n)}{2\ln n}\right.$$

$$\left.+\theta\left(\frac{1}{2\ln n}\right)\right)$$

$$u_n=(2\ln n)^{\frac{1}{2}}\cdot\frac{x}{2\ln n}+\left((2\ln n)^{\frac{1}{2}}-\right.$$

$$\left.-(2\ln n)\frac{\frac{1}{2}\ln 4\pi+\frac{1}{2}\ln(\ln n)}{2\ln n}+\theta\left((2\ln n)^{\frac{1}{2}}\frac{1}{2\ln n}\right)\right)$$

$$u_n=\frac{x}{(2\ln n)^{1/2}}+\left((2\ln n)^{\frac{1}{2}}-\right.$$

$$\left.-\frac{1}{2}\frac{\ln 4\pi+\ln(\ln n)}{(2\ln n)^{1/2}}+\theta\left(\frac{1}{(2\ln n)^{1/2}}\right)\right)$$

$$u_n=\frac{x}{a_n}+b_n+\theta\left(a_n^{-1}\right).$$

From (2.2) we have

$P(M_n\le u_n)\to\exp(-\mathrm{e}^{-x})$ where $\tau=\mathrm{e}^{-x}$

$P\left\{M_n\le\frac{x}{a_n}+b_n+\theta\left(a_n^{-1}\right)\right\}\to\exp\left(-\mathrm{e}^{-x}\right)$ or

$P\{a_n(M_n-b_n)+\theta(1)\le x\}\to\exp\left(-\mathrm{e}^{-x}\right)$ that is (2.5) Q.E.D.

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