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THE FIRST ORDER DERIVATIVE OF FUNCTION $\chi_i(t)$

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Abstract: *Linear partial differential equations of second order hyperbolic type are involved in describing oscillating and wave processes in elastic and electromagnetic mediums. In this paper we study two equations, from the canonical form [1] of equation of hyperbolic type, and using the method of integration along the corresponding characteristics, we obtain the first order derivative of* $\chi_i(t)$ *with respect to t.*

Key-Words: *wave equation, hyperbolic equations, characteristics.*
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1. INTRODUCTION

Linear partial differential equations of second order hyperbolic type have a wide range of applications in mathematical physics problems. They appear in the description of oscillatory and wave processes in elastic and electromagnetic environments. Certain types of first order hyperbolic systems can be reduced to wave equation [2]. But a second order hyperbolic equation, in turn, can be treated as a first order hyperbolic system.

Let consider the set

$$\Omega_T = \{(x,t) : x \in R,\ 0 \le t \le T\}, \tag{1}$$

and the second order hyperbolic equation

$$u_{tt} = a^2 u_{xx} + bu_x + cu_t + du + F\ . \tag{2}$$

with the Cauchy data

$$\begin{cases} u(x,0) = \varphi(x), & x \in \mathbf{R}, \\ u_t(x,0) = \psi(x), & x \in \mathbf{R}. \end{cases} \tag{3}$$

The subsidiary information about the problem (2)-(3) solutions is

$$u(x_i,t) = \chi_i(t),\ \ 0 \le t \le T\ ,\ \ \ 0 \le i \le n\ . \tag{4}$$

We wish to find a function from (2)-(4) with the form $F(x,t) = \sum_{i=1}^{n} g_i(x,t)p_i(t) + h(x,t)$ where $g_i(x,t)$ and $h(x,t)$ are the known functions, while the unknown functions $p_i(t)$, $1 \le i \le n$, are sought. Let $x_1 < x_2 < \ldots < x_{n-1} < x_n$ and let the function $a(x,t)$ be positive, bounded and twice continuously differentiable. The equation of the characteristics $\xi = \xi_i(\tau; x,t)$ passing through a point (x,t) is

$$\begin{cases} \dfrac{d\xi_i}{d\tau} = \varepsilon_i a(\xi_i, \tau), \\ \xi_i(t; x, t) = x, \end{cases} \tag{5}$$

where $i = 1,2;\ \varepsilon_1 = -1$ şi $\varepsilon_2 = 1$.

2. PROBLEM FORMULATION

Let us consider second-order hyperbolic equation where the coefficients $b(x,t) = 0$ and $c(x,t) = 0$ are taken zero. In these circumstances we can establish the following result [1]:

The equation

$$u_{tt} = a^2 u_{xx} + du + F \tag{6}$$

is equivalent with the system

$$r_t = Kr_x + Dr + \Phi, \tag{7}$$

where the new unknowns are given by the the following relations:

$$r_1 = u,\ r_2 = \frac{1}{2}\left(\frac{v}{a} + w\right),\ r_3 = \frac{1}{2}\left(\frac{v}{a} - w\right), \tag{8}$$

and

$$r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix},$$

$$D = \frac{1}{2a}\begin{pmatrix} 0 & 2a^2 & 2a^2 \\ d & aa_x - a_t & aa_x - a_t \\ d & -aa_x - a_t & -aa_x - a_t \end{pmatrix},$$

$$\Phi = \frac{1}{2a}\begin{pmatrix} 0 \\ F \\ F \end{pmatrix}. \tag{9}$$

3. PROBLEM SOLUTION

Lemma. Let the system $r_t = Kr_x + Dr + \Phi$ and the relationships $u(x_i,t) = \chi_i(t)$, $0 \le t \le T$, $0 \le i \le n$. If $v = u_t$, by integrating the last two equations of the system along the corresponding characteristics, then the $\chi_i'(t)$ will take the form

$$\chi_i'(t) = R(x_i,t) + a(x_i,t) \times \left[\int_{L_1(x_i,t)} [Bu + Cv + H\mathbf{gp}] + \right.$$

$$\left. + \int_{L_2(x_i,t)} [Bu + Ev + H\mathbf{gp}] \right] \tag{10}$$

where

$$B = \frac{d}{2a},\ C = \frac{aa_x - a_t}{2a^2},\ E = \frac{-aa_x - a_t}{2a^2},$$

$$H = \frac{1}{2a},\ \mathbf{p} = (p_1, ..., p_n),\quad \mathbf{g} = (g_1, ..., g_n),$$

$$\mathbf{pg} = \sum_{i=1}^{n} p_i g_i. \tag{11}$$

Proof. From $u(x_i,t) = \chi_i(t)$, $0 \le t \le T$, and given that $u_t = v$ we get relations $v(x_i,t) = u_t(x_i,t) = \chi_i'(t)$, $0 \le i \le n$. But $v = a(r_2 + r_3)$, that is

$$v(x_i,t) = a(x_i,t)[r_2(x_i,t) + r_3(x_i,t)], \tag{12}$$

$$v(x_i,t) = \chi_i'(t). \tag{13}$$

so

$$a(x_i,t)[r_2(x_i,t) + r_3(x_i,t)] = \chi_i'(t),\ 0 \le i \le n. \tag{14}$$

Integrating equality $u_t = v$ results:

$$u(x,t) = \varphi(x) + \int_0^t v(x,\tau)d\tau. \tag{15}$$

We integrate the second equation of system (7), along the corresponding characteristic, as follows:

$$(r_2)_t = a(r_2)_x + \frac{1}{2a}[dr_1 + (aa_x - a_t)(r_2 + r_3)] + \frac{F}{2a}. \tag{16}$$

but

$$\begin{cases} r_1 = u \\ r_2 + r_3 = \dfrac{v}{a}, \end{cases}$$

$$F(x,t) = \sum_{i=1}^{n} g_i(x,t)p_i(t) + h(x,t), \tag{17}$$

so

$$(r_2)_t = a(r_2)_x + \frac{d}{2a}u + \frac{(aa_x - a_t)}{2a^2}v + \frac{1}{2a}\mathbf{gp} + \frac{h}{2a}. \quad (18)$$

Using the integral along the characteristic

$$\int_{L_i(x,t)} (\varphi) = \int_0^t \varphi(\xi_i(\tau;x,t),\tau)d\tau \text{ for } i=1,$$

we obtain

$$\int_{L_1(x,t)} (r_2)_t = \int_{L_1(x,t)} \left[a(r_2)_x + \frac{d \cdot u}{2a} + \frac{(aa_x - a_t)v}{2a^2} + \frac{\mathbf{gp} + h}{2a} \right] \quad (19)$$

$$\int_{L_1(x,t)} (r_2)_t = \int_{L_1(x,t)} [a(r_2)_x] + \int_{L_1(x,t)} \left[\frac{d}{2a}u + \frac{(aa_x - a_t)}{2a^2}v + \frac{1}{2a}\mathbf{gp} + \frac{h}{2a} \right] \quad (20)$$

$$\int_{L_1(x,t)} [(r_2)_t - a(r_2)_x] = \int_0^t \left[\frac{\partial r_2}{\partial t}(\xi_1(\tau;x,t),\tau) - a(\xi_1(\tau;x,t),\tau)\frac{\partial r_2}{\partial x}(\xi_1(\tau;x,t),\tau) \right] d\tau = \int_0^t \left[\frac{d}{dt} r_2(\xi_1(\tau;x,t),\tau) \right] d\tau = r_2(\xi_1(t;x,t),t) - r_2(\xi_1(0;x,t),0) \quad (21)$$

$$r_2(x,t) = r_2(\xi_1,0) + \int_{L_1(x,t)} \left[\frac{d}{2a}u + \frac{(aa_x - a_t)}{2a^2}v + \frac{1}{2a}\mathbf{gp} + \frac{h}{2a} \right] \quad (22)$$

Integrate the third equation of system (7), along the corresponding characteristic using the similar calculations, we have:

$$r_3(x,t) = r_3(\xi_2,0) + \int_{L_2(x,t)} \left[\frac{d}{2a}u - \frac{(aa_x + a_t)}{2a^2}v + \frac{1}{2a}\mathbf{gp} + \frac{h}{2a} \right] \quad (23)$$

Make the following notations

$$B = \frac{d}{2a}, \quad C = \frac{aa_x - a_t}{2a^2}, \quad E = \frac{-aa_x - a_t}{2a^2}, \quad H = \frac{1}{2a}, \quad \mathbf{p} = (p_1, \ldots, p_n), \quad \mathbf{g} = (g_1, \ldots, g_n), \quad \mathbf{pg} = \sum_{i=1}^{n} p_i g_i. \quad (24)$$

and obtain

$$r_2 = r_2(\xi_1,0) + \int_{L_1(x,t)} [Bu + Cv + H\mathbf{gp}] + \int_{L_1(x,t)} \frac{h}{2a} \quad (25)$$

$$r_3 = r_3(\xi_2,0) + \int_{L_2(x,t)} [Bu + Ev + H\mathbf{gp}] + \int_{L_2(x,t)} \frac{h}{2a} \quad (26)$$

Adding the relations (25) and (26) and multiplying the result by $a(x,t)$ we obtain:

$$a(r_2 + r_3) = a[r_2(\xi_1,0) + r_3(\xi_2,0)] + a \times \left[\int_{L_1(x,t)} [Bu + Cv + H\mathbf{gp}] + \int_{L_2(x,t)} [Bu + Ev + H\mathbf{gp}] \right] + a\left(\int_{L_1(x,t)} \frac{h}{2a} + \int_{L_2(x,t)} \frac{h}{2a} \right). \quad (27)$$

We now consider the relationship $a(r_2 + r_3) = v$ and get to the equation:

$$v(x,t) = R(x,t) + +a \times \\ \times \left[\int_{L_1(x,t)} [Bu + Cv + H\mathbf{gp}] + \int_{L_2(x,t)} [Bu + Ev + H\mathbf{gp}] \right] \tag{28}$$

where

$$R(x,t) = a[r_2(\xi_1, 0) + r_3(\xi_2, 0)] + \\ + a\left(\int_{L_1(x,t)} \frac{h}{2a} + \int_{L_2(x,t)} \frac{h}{2a} \right) \tag{29}$$

which is a known function, because:

$$r_2(x,0) = \frac{1}{2}\left(\frac{v(x,0)}{a(x,0)} + w(x,0) \right) = \frac{1}{2}\left(\frac{\varphi(x)}{a} + \varphi'(x) \right)$$

$$r_3(x,0) = \frac{1}{2}\left(\frac{v(x,0)}{a(x,0)} - w(x,0) \right) = \frac{1}{2}\left(\frac{\varphi(x)}{a} - \varphi'(x) \right)$$

We know that $v(x_i, t) = \chi_i'(t)$, $0 \le i \le n$, so the remaining equations are obtained from equation (28) to order $x = x_i$, $1 \le i \le n$.

Thus:

$$\chi_i'(t) = R(x_i, t) + a(x_i, t) \times \\ \times \left[\int_{L_1(x_i,t)} [Bu + Cv + H\mathbf{gp}] + \int_{L_2(x_i,t)} [Bu + Ev + H\mathbf{gp}] \right] \tag{30}$$

4. CONCLUSIONS

In this article we found the first order derivative of $\chi_i(t)$ with respect to *t*, integrating along the corresponding characteristics the last two equations of the canonical form of a second order hyperbolic equation. This derivative is useful in proving the theorem of existence and uniqueness of the solution of inverse problem (2)-(4).

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