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UAV CONTROLLER SYNTHESIS USING LQ–BASED DESIGN METHODS

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Abstract: *The LQ (Linear Quadratic) based design methods are powerful tools for control system controllers' synthesis and they are applied since many decades for control system design purposes. The model-based LQ methods may be divided into two different methods. First of them is the LQR (Linear Quadratic Regulator) method. This design method supposes available for the measurements state variables of the control system. This problem is the so-called "full state feedback" problem. The second LQ design method is the LQG (Linear Quadratic Gaussian) one. This method allows consideration of the influence of internal and external stochastic disturbances affecting motion of the aircraft. The purpose of the author is to summarize the theoretical backgrounds of design methods listed above and to show design examples for solution of LQR and LQG model-based design methods applied to synthesize controller for the Unmanned Aerial Vehicle (UAV) system.*

Keywords: *military robots, recce surface robots, air robot systems, CAD.*

1. INTRODUCTION

LQ based design methods are widely applied for optimal control of aircraft. The LQR method allows determining optimal control law minimizing pre-defined integral performance criteria. The LQG design method allows consideration of simultaneous external and internal disturbances affecting motion of the aircraft.

Modern control systems are analyzed in [1, 3, 4, 5, 6, 10, 11, 13, 14]. LQR design problem in the focus of attention of references of [1, 2, 3, 4, 10, 11, 13, 14]. The LQG design method is outlined with applications in [5, 6, 13]. A special attention must be paid to applications of the proposed LQR and LQG methods applied in design of the automatic flight control systems [8, 9]. Pokorádi in [12] gave full-scale description of derivation of dynamic systems' mathematical models, and signals applied for system analysis purposes.

Computer aided analysis and design of the dynamical systems are supported by computer packages like MATLAB® supplemented with appropriate toolboxes applied at our Department of Military Robotics, at Zrínyi Miklós National Defense University [2, 7].

2. THE LQR DESIGN PROBLEM FORMULATION

Dynamics of the LTI system may be defined using the following state and output equations [1, 3, 4, 5, 6, 8, 9, 11]:

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Bu},\ \mathbf{y}=\mathbf{Cx}+\mathbf{Du}. \qquad (2.1)$$

The block diagram of the closed loop system – for $\mathbf{D}=0$ – built by equations (2.1) may be seen in Figure 1.

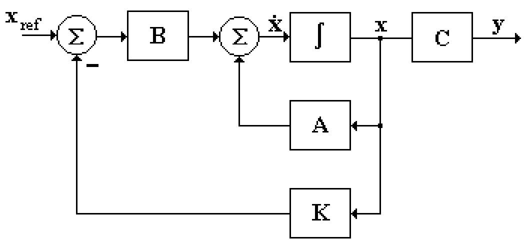

Figure 1. Block Diagram of the Control System

Optimal control law may be determined evaluating the following integral performance criteria [3, 4, 5, 8, 10, 13, 14]:

$$J = \frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right) dt \to Min\,. \qquad (2.2)$$

In cost function of equation (2.2) main design parameters are weights of $\mathbf{Q}\geq 0$ and weights of $\mathbf{R}>0$. If weighting matrix **Q** is very large relative to weighting matrix **R** one may get a closed loop system response with large overshoots. If weighting matrix **R** is chosen to be very large relative to **Q** control system has smaller actuators, electric motors, amplifier gains and other devices. During controller synthesis weighting matrices may be derived using the so-called inverse square rule.

The LQ optimal control problem may be solved using wide variety of techniques. Let us consider method of Euler-Lagrange equations, Hamilton-Jacobi-Bellman theory and Pontriagin's minimum principle. Firstly, let us define the so-called Hamiltonian matrix to as follows below [1, 2, 3, 4, 8, 11, 13, 14]:

$$\mathbf{H}(\mathbf{x},\lambda,t) = \frac{1}{2}\left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x}^{\mathrm{T}} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}^{\mathrm{T}}\right) + \lambda^{\mathrm{T}}\left(\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}\right), \qquad (2.3)$$

where λ is the Lagrange multiplier.

It is well-known that Pontriagin's minimum principle states that optimal state and control trajectories must satisfy the following equations [10, 11, 13, 14]:

$$\frac{\partial \mathbf{H}}{\partial \lambda} = \dot{\mathbf{x}};\ \frac{\partial \mathbf{H}}{\partial \mathbf{x}} = -\dot{\lambda};\ \frac{\partial \mathbf{H}}{\partial \mathbf{u}} = 0\,. \qquad (2.4)$$

Using rules for differentiation of matrices and vectors equations (2.4) may be rewritten in the following manner

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{x}(0) = \mathbf{x}_O\,, \qquad (2.5)$$

$$-\dot{\lambda} = \mathbf{Q}\mathbf{x} + \mathbf{A}^{\mathrm{T}}\lambda,\ \lambda(T) = 0\,, \qquad (2.6)$$

$$\mathbf{u}^{O} = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\lambda\,. \qquad (2.7)$$

Equation (2.7) defines the optimal control law. The coupled equations (2.5), (2.6) and (2.7) may be regarded as the 'two point boundary value problem' (TPBWP). Substituting equation of control law (2.7) into state equation (2.5) results in following formula [1, 2, 3, 4, 8, 11, 13, 14]:

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\\ -\mathbf{Q} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \lambda\end{bmatrix} \hat{=} \mathbf{H}\begin{bmatrix}\mathbf{x}\\ \lambda\end{bmatrix}. \qquad (2.8)$$

Let us make the following substitution in equation (2.8):

$$\lambda = \mathbf{P}\mathbf{x}, \qquad (2.9)$$

where **P** is the so-called cost matrix.

Differentiating equation (2.9) with respect to time, and considering equations (2.5) and (2.7) following equation may be derived:

$$\frac{d\lambda}{dt} = \frac{d\mathbf{P}}{dt}\mathbf{x} + \mathbf{P}\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{P}}{dt}\mathbf{x} + \mathbf{P}\mathbf{A}\mathbf{x} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x} = -\mathbf{Q}\mathbf{x} - \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{x}\,. \qquad (2.10)$$

The sufficient condition for optimal control is that **P** must satisfy the following Ricatti differential equation [11, 13, 14]:

$$-\frac{d\mathbf{P}}{dt} = \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P},\ \mathbf{P}(\mathrm{T}) = 0 \qquad (2.11)$$

Solution of the optimal controller synthesis problem using Ricatti-equation in control theory is regarded as the finite time problem. This solution results in the linear time varying controller of the feedback [1, 2, 3, 4]:

$$\mathbf{u}^{o}(t) = -\mathbf{K}(t)\mathbf{x}(t),\ \mathbf{K}(t) = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}(t)\,. \qquad (2.12)$$

Equation (2.11) is a nonlinear, first order differential equation, which has to be solved backwards in time [1, 2, 3, 4]. During solution of the infinite time LQR problem it is considered that $\mathrm{T}\to\infty$.

It is obvious that under mild conditions cost matrix **P** may be considered as constant and, solution of Ricatti-equation results in the asymptotically stable closed loop control system.

In this particular case, equation (2.11) may be rewritten as:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} = 0\,, \qquad (2.13)$$

And, optimal control vector may be derived as:

$$\mathbf{u}^{o}(\mathrm{t}) = -\mathbf{K}\mathbf{x}(t),\ \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\,. \qquad (2.14)$$

Equation (2.13) is known as algebraic Ricatti equation (ARE). Conditions defined by equations (2.13) and (2.14) are necessary and sufficient for existence of the optimal controller, which will asymptotically stabilize the control system.

The steps of optimal control law synthesis includes following two steps:

1, solution of the ARE - equation (2.13) - in order to find the constant cost matrix **P**,

2, substituting cost matrix **P** into equation (2.14).

The resulting feedback gain matrix **K** is an optimal for the given set of weightings of those matrices of **Q** and **R** of the performance integral criteria [1, 2, 3, 4, 8, 11, 13, 14].

3. OPTIMAL CONTROL LAW SYNTHESIS USING LQG DESIGN METHOD

The LQ based control law synthesis problem is solved in the time domain (LQR problem) and in the frequency domain (LQG problem). The LQR problem is for the determination of the optimal control law when all state variables are available for measurement. This case is rarely may be met in the practice. The more realistic case is the LQG problem, which is representation of the output feedback problem.

During solution of the LQG controller synthesis problem there is considered the disturbed state-space model of the plant as given below [1, 2, 3]:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \Gamma\mathbf{w}, \;\; \mathbf{y} = \mathbf{C}\mathbf{x} + \upsilon \qquad (3.1)$$

In equation (3.1), **x** is state vector, **u** is the control vector, **y** is the vector of measured outputs, **w** is the random plant disturbance, and υ is the random sensor noise. Both disturbances are uncorrelated, Gaussian stationary random processes with zero-mean values. Finally, **Γ** is the input matrix for the external disturbance of $\mathbf{w}$. Covariances of the random processes are listed below [5, 6, 13]:

$$\begin{aligned} &E\left\{\mathbf{w}\mathbf{w}^T\right\} = \mathbf{Q}_0 \geq 0 \\ &E\left\{\upsilon\upsilon^T\right\} = \mathbf{R}_0 > 0 \quad , \\ &E\left\{\mathbf{w}\upsilon^T\right\} = 0 \end{aligned} \qquad (3.2)$$

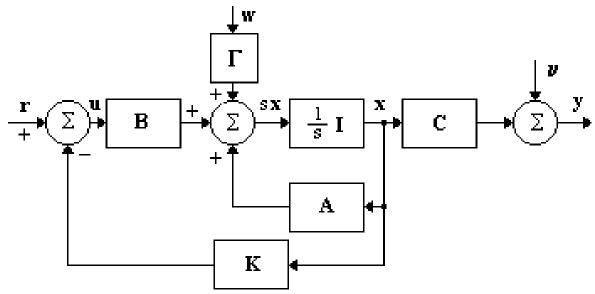

Figure 2. Block Diagram of the Dynamic System

Block diagram of the dynamic system defined by equation (3.1) may be seen in Fig. 2.

From Figure 1 the optimal control law – for **r** = 0 - may be derived as follows below:

$$\mathbf{u} = -\mathbf{K}\mathbf{x}\,. \qquad (3.3)$$

The optimal state feedback gain matrix is as given below

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\,. \qquad (3.4)$$

The positive definite cost` matrix **P** may be found solving the algebraic Ricatti-equation [1, 2, 3, 4, 7, 8, 11, 13, 14]:

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} = 0\,. \qquad (3.5)$$

The synthesis of the LQG controller may be achieved using the so-called separation principle.

The derived control law will minimize the following average integral 'cost' function, i.e.

$$J = \lim_{T\to\infty} E\left\{\int_0^T \left(\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\,\mathbf{u}\right)dt\right\} \to Min\,. \qquad (3.6)$$

Using the separation principle the control law synthesis problem may be solved in two separate stages.

Firstly, the so-called deterministic cost is minimized solving the reduced-matrix Ricatti equation. The Kalman-filter state equation may be derived as given below [1, 2, 3, 4, 7, 8, 11, 13, 14]:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\left(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}\right). \qquad (3.7)$$

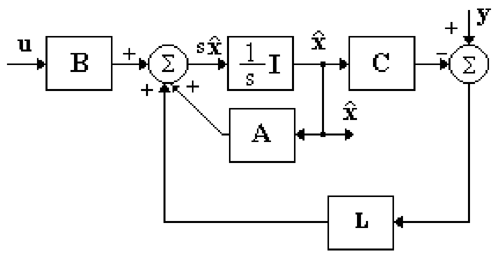

Figure 3. Block Diagram of the State Observer

The block diagram of the state observer – Kalman-filter – derived by equation (3.7) may be seen in Figure 3.

The input of the state observer are **u** and **y**, the output is the estimate $\hat{\mathbf{x}}$. The static gain of the optimal state observer **L** may be found by equation given below

$$\mathbf{L} = \mathbf{\Sigma C}^{\mathrm{T}} \mathbf{R}_0^{-1} . \quad (3.8)$$

In equation (3.8) **L** is the Kalman-filter static gain, **Σ** is a positive definite cost matrix and, $\mathbf{R_0}$, $\mathbf{Q_0}$ is the set of weighting matrices of the state and the input vectors, respectively.

The cost matrix **Σ** may be derived solving the following equation [2, 3, 4, 8, 11, 13, 14]:

$$\mathbf{A\Sigma} + \mathbf{\Sigma A}^{\mathrm{T}} - \mathbf{\Sigma C}^{\mathrm{T}} \mathbf{R}_0^{-1} \mathbf{C\Sigma} + \mathbf{\Gamma Q}_0 \mathbf{\Gamma}^{\mathrm{T}} = 0 . \quad (3.9)$$

The structure of the LQG compensator may be derived as the series connection of the Kalman-filter with the state feedback gain matrix.

The block diagram of control system may be seen in Figure 4.

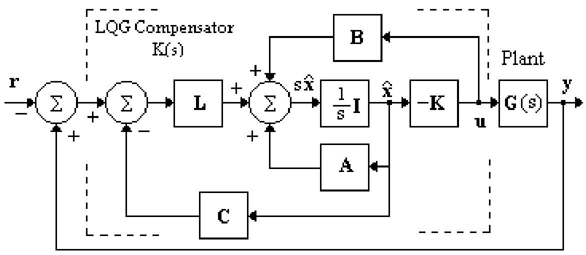

Figure 4. The Structure of the LQG Compensator

This representation of the control system may be applied if the linear plant model is the reliable one, and there are well-defined stochastic processes to be considered. In this particular case the concern is to minimize the cost function defined by equation (3.6).

During solution of the LQG controller synthesis problem weighting matrices **Q** and **R** – used for Linear Quadratic Regulator design stage – and weighting matrices $\mathbf{Q}_o$ and $\mathbf{R}_o$ – used for Linear Quadratic Estimator design phase are applied as tuning parameters [1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14].

4. CONCLUSIONS, FUTURE WORK

Optimal control law synthesis technique is widely applied in design of dynamic systems controllers. The design process is supported by large scale references, and also many application examples are available.

The LQG method is more realistic than the LQR one. These methods may be applied for preliminary design of the automatic flight control systems. The model-based design supposes existence of the identified model of the aircraft.

Methods are based on preliminary settings, and further heuristic settling of the weights in integral criteria defined by equation (2.2).

Application of these methods predicts high level of theoretical knowledge, and experiences in working with heuristic setting of weights in solution of optimal control systems.

The paper is proposed to use in design of optimal dynamic systems, namely for design of the automatic flight control systems of the UAV.

REFERENCES

1. Burns, R. S. *Advanced Control Engineering*: Butterworth-Heinemann, Oxford-Auckland-Boston-Johannesburg-Melbourne-New Delhi (2001).

2. Control System Toolbox 8.0, *Getting Started Guide*: The MathWorks™, Inc. (2008).

3. Dorf, R. C. – Bishop, R. H., *Modern Control Systems*: Prentice Hall International, Upper Saddle River, New Jersey (2001).

4. Franklin, G. F. – Powell, J. D. – Emami-Naeini, A. *Feedback Control of Dynamic Systems*: Prentice-Hall, Pearson Education International (2002).

5. Gimble, M.J., *Robust Industrial Control Optimal - Design Approach for Polinomial Systems*: Prentice-Hall, Inc. (1994).

6. Maciejowski, J. M. *Multivariable Feedback Design*: Addison-Wesley Publishing Company (1989).

7. *MATLAB® 7 (R2010b), Getting Started Guide*, The MathWorks™, Inc., (2009).

8. McLean, D., *Automatic Flight Control Systems*: Prentice-Hall International, Inc.,: New York-London-Toronto-Sydney-Tokyo-Singapore (1990).

9 Nelson, L. C., *Flight Stability and Control*: McGraw-Hill Companies, Inc., Boston, Massachusetts, Burr Ridge (1998).

10. Nise, N. S. *Control Systems Engineering*, John Wiley & Sons, Inc., (2004).

11. Ogata, K. *Modern Control Engineering*: Prentice-Hall International, Inc. (1990).

12. Pokorádi, L. *Modelling systems and processes*: Campus Kiadó, Debrecen (2008).

13. Shahian, B. – Hassul, M. *Control System Design Using MATLAB®*: Prentice-Hall, Englewood Cliffs, New Jersey (1993). l,

14. Stefani, R. T. – Shahian, B. – Savant Jr., C. J. – Hostetter, G. H. *Design of Feedback Control Systems*: Oxford University Press, New York-Oxford (2002).