SOLVING THE BILEVEL LINEAR PROGRAMMING PROBLEM USING THE MONTE CARLO METHOD

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Abstract: *In this paper we propose to use the "Monte Carlo" method for solving bilevel linear programming (the bilevel linear programming problem – BLP problem). In the BLP problem, each decision maker tries to optimize their own objective function without considering the objective of the other party, but the decision of each party affects the objective value of the other party as the decision space. The existing methods for solving the BLP problem can be grouped into four categories: a) methods based on vertices enumeration; b) methods based on Kuhn-Tuck conditions; c) the fuzzy approach; d) metaheuristics methods. Starting from Gnedenko's theorem, this paper uses the "Monte Carlo" method to determine an approximate solution for the BLP problem. The numerical example presents the performance of the proposed approach.*

Keywords: *vertices enumeration method, decision makers, random search method, uniform random numbers, relative error*

1. BILEVEL LINEAR PROGRAMMING PROBLEM FORMULATION

The bilevel programming problem has its origins in the work of Stackelberg „The Theory of the Market Economy (Oxford University Press”, 1952). The BLP arises in paper of Bracken and McGill "Mathematical Programs with Optimization Problems in the Constraints" (Operations Research Vol. 21 No. 1, 1973) [1]. Although the origins of BLP problem are in the game theory, BLP problem has applications in many technical fields and to set an example in hydrology to efficient use of water resources [3].

Many references can be found in the report of Vicente and Calamai, Bilevel and Multilevel Programming: A Bibliography Review [8].

The general formulation of a bilevel programming problem is as follows [2]:

(upper level)

$$\begin{cases} \min\limits_{x,y} F(x,y) \\ s.t.\ G(x,y) \le 0 \end{cases} \tag{1}$$

where y can be solved from

(lower level)

$$\begin{cases} \min\limits_{y} f(x,y) \\ s.t.\ g(x,y) \le 0 \end{cases} \tag{2}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$. The variables of problem (1) , $x \in R^{n_1}$ are considered upper level and $y \in R^{n_2}$ are the lower level variables. The upper level decision maker controls over vector x, and the lower level decision maker controls over vector y.

In the following we consider the particular case when the objective functions and the constraints are linear functions, obtaining the BLP problem.

In the BLP problem, each decision maker tries to optimize its own objective function without considering the objective of the other party, but the decision of each party affects the objective value of the other party as well as the decision space. The linear BLP problem is an optimization model formulated as follows:

$$\begin{cases} \min\limits_{x} F(x,y) = c_1 x + d_1 y \\ \min\limits_{y} f(x,y) = c_2 x + d_2 y \\ s.t. \qquad A_1 x + A_2 y \le b, \\ \qquad\quad x,\ y \ge 0 \end{cases} \tag{3}$$

where: $F(x,y)$ is the objective function of the leader and $f(x,y)$ is the objective function of the follower. Also, $x \in R^{n_1}$ is a vector controlled by leader and $y \in R^{n_2}$ is a vector by follower, $A_1 \in \mathcal{M}_{m,n_1}(R)$ is a $(m \times n_1)$ real matrix and $A_2 \in \mathcal{M}_{m,n_2}(R)$ is a $(m \times n_2)$ real matrix. Here $x \ge 0$ means $x_i \ge 0$, $i = \overline{1,n_1}$.

The leader decision is priority and the leader gets feedback from the follower. There are many methods to solve this kind of problems; the existing methods can be grouped into the following categories [4]:

(a) Methods based on vertices enumeration: the optimal solution should be in vertex points belonging in feasible space determined by constraints [2],

(b) Methods Based on Kuhn-Tucker conditions: the BLP problem becoming a one-level problem replacing the second level problem with complementary constraints [9]

(c) The fuzzy approach: the objective functions of the leader or the follower or both objective functions are considered like a memberships function [6], and

(d) Methods based on meta heuristics: to solve the BLP problem can be used genetic algorithm [4,5], algorithm based on simulated annealing etc.

Our proposed approach can be included in last category of methods to solve BLP problem.

2. THE PROPOSED METHOD: OPTIMIZATION USING RANDOM SEARCH

In this section we present methods of optimization based on *random search.* The problem is the following, $\min\limits_{x \in D} f(x)$, where $D \subset R^k$ is a $k-$dimensional set. There is a vast variety of professional literature based on properties of $f(x)$ or of D. Here we are interested in determining the point $x^* \in D$ such as $\min\limits_{x \in D} f(x) = f(x^*) = f^*$, i.e. x^* is a *global* minimum point. The global minimum point is selected from several local minimums.

The idea of *random search* is the following [7]:

Generate a large number N of random points $X_1, X_2, ..., X_N$, uniformly distributed in D (supposed to be a bounded set). Then calculate $f(X_i)$, $1 \le i \le n$ and take $f^*_{(N)} = f\left(X^*_{(N)}\right) = \min\limits_{1 \le i \le N} f(X_i)$.

A theorem of Gnedenko (1943) says that in some conditions (i.e. f is a continuous function), we have $\lim_{N\to\infty} f^*_{(N)} = f^*$, $\lim_{N\to\infty} X^*_{(N)} = x^*$. If the optimum solution $x^* \in T^* \subset D$, (T^* is a capture set containing the solution), then it is known that for a given risk ε, $0<\varepsilon<1$ there is a p such as [7]:

$$P\left(X_N \in T^*; \left\|X_N - x^*\right\| \geq \varepsilon\right) = p,$$

it is necessary to use

$$N > \left[\frac{\log\varepsilon}{\log(1-p)} + 1\right] = N^* , \tag{4}$$

the previous probability being calculated for the assumed distribution of X. As p is not known, one can use $p > 1-\varepsilon$. The following algorithm allows the approximation of $x^* \in D$.

The Randseach Algorithm

Step 0. Input: $N-$ number of points uniformly distributed on D used for determining optimum value of leader function , X^*

$M-$ number of points extracted from the N used for determining optimum value of follower function, Z^*

$K-$number of points uniformly distributed on the segment $\left[X^*, Z^*\right]$

Initialize: $i \leftarrow 1$; $j \leftarrow 0$;

Step 1. While $i \leq N$ execute

generate $x \sim> \mathcal{U}(D)$

calculate $C_{i,1} = F(x)$

for $j = \overline{2, k+1}$ $C_{i,j} = x_{j-1}$

endfor

endwhile

Step 2. Sort C after the first column in C

for $j = \overline{2, k+1}$

$x^*_{1,j-1} = C_{1,j}$

endfor

Retain M lines of C in B

for $i = \overline{1, M}$ calculated

$B_{i,1} = f\left(C_{i,2}, C_{i,3}, ..., C_{i,k}\right)$

for $j = \overline{2, k+1}$ $B_{i,j} = x_{j-1}$

endfor

endfor

Sort B after the first column in B

for $j = \overline{2, k+1}$ $y^*_{1,j-1} = B_{1,j}$

endfor

for $i = \overline{1, K}$

Generate $x \sim> \mathcal{U}\left(\left[x^*, y^*\right]\right)$

Calculate $F(x)$, $f(x)$.

$S_{i,1} = F(x)$

$S_{i,k+2} = f(x)$

for $j = \overline{2, k+1}$ $\quad x^{**}_{i,j-1} = S_{i,j}$

endfor

endfor

Sort S after the first column in S;

Step 3. Delivre $F_{opt} = F(x^{**})$, $f_{opt} = f(x^{**})$, obtained in x^{**}

Stop!

Remark: The matrix S have on the first line and first column the optimum value for leader objective function, on the first line and last column the optimum value for follower objective function and on the interior columns of the first line, x^{**}, are the coordinates of approximate optimal point!

3. A NUMERICAL EXAMPLE

We will customize the algorithm proposed in the previous section for the case of k=3 and we will compare the solution obtained with the help of the polyhedron vertex enumeration method with the approximate solution obtained with the proposed method. At the end of the section we will determine the relative error and we will make the necessary considerations.

Consider the following problem:

$$\min_{(x,y)} F(x,y,z) = 3x + y + z$$

$$\min_{z} f(x,y,z) = 2x - y - 7z$$

$$\text{restrictions} \begin{cases} 2x - y + 2z \geq 3 \\ x - y + z \leq 1 \\ x - y + 2z \leq 2 \\ y - z \leq 1 \end{cases}$$

These planes define an $ABCD$ tetrahedron with the following vertices:
$A(1,1,1)$, $B(2,1,0)$ $C(1,3,2)$, $D(2,2,1)$.

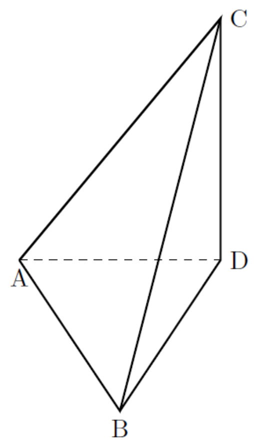

FIG. 1 Tetrahedron of restrictions

3.1 Enumeration Method

Firstly we use the enumeration vertices method to find exact solution.

$F(A)=F(1,1,1)=3\cdot 1+1+1=5$, $F(B)=F(2,1,0)=3\cdot 2+1+0=7$,

$F(C)=F(1,3,2)=3\cdot 1+3+2=8$, $F(D)=F(2,2,1)=3\cdot 2+2+1=9$,

$\min\limits_{t\in\{A,B,C,D\}}\{F(t)\}$ is obtained in the point *A*.

We have:

$$m=\max\left\{\frac{3}{2}-x+\frac{y}{2},y-1,0\right\}\le z\le \min\left\{1-x+y,1-\frac{x}{2}+\frac{y}{2}\right\}=M$$

For that $[m, M]\neq\Phi$ it needs that $m\le M$, which is equivalent with the next inequalities

$$\frac{3}{2}-x+\frac{y}{2}<1-x+y\Rightarrow y\ge 1-x+\frac{y}{2}<1-x+y\Rightarrow y\ge 1 \quad (5)$$

$$\frac{3}{2}-x+\frac{y}{2}<1-\frac{x}{2}+\frac{y}{2}\Rightarrow x\ge 1 \quad (6)$$

$$y-1\le 1-x+y\Rightarrow x\le 2 \quad (7)$$

$$y-1\le 1-\frac{x}{2}+\frac{y}{2}\Rightarrow x+y\le 4 \quad (8)$$

$$0\le 1-x+y\Rightarrow x-y\le 1 \quad (9)$$

$$0\le 1+\frac{x}{2}+\frac{y}{2}\Rightarrow x-y\le 2 \quad (10)$$

The (10) relation is superfluous because it results from the relation (9). The (9) relation is superfluous because it results from the (5) and (7) relations. We can note that in the domain described by the above inequalities we obtain from the (5) relation, $y-1\ge 0$, so we can discard the zero from the description of *m*

$$1-x+y\le 1-\frac{x}{2}+\frac{y}{2}\Rightarrow y\le x$$

$$\frac{3}{2}-x+\frac{y}{2}\ge y-1$$

$$\frac{5}{2}\ge x+\frac{y}{2}$$

$$5\ge 2x+y$$

We have $z^*=M$ because *f* is decreasing in the variable *z*.

$z^*=M=1-x+y$ in the triangle with the vertices $A'(1,1)$, $B'(2,1)$, $D'(2,2)$

$$F(x,y,z^*)=3x+y+1-x+y=2x+2y+1$$

$$(1,1)\to 2\cdot 1+2\cdot 1+1=5$$

$$(2,1)\to 2\cdot 2+2\cdot 1+1=7$$

$$(2,2)\to 2\cdot 2+2\cdot 2+1=9$$

considering $A'(1,1)$, $B'(2,1)$, $D'(2,2)$, where A', B', D' are the projections of points, A',B',D' on the XOY plane.

$$F(x,y,z^*)=3x+y+1-\frac{x}{2}+\frac{y}{2}=\frac{5}{2}x+\frac{3}{2}y+1$$

$$(1,1)\to\frac{5}{2}\cdot 1+\frac{3}{2}\cdot 1+1=5$$

$(1,3) \to \frac{5}{2} \cdot 1 + \frac{3}{2} \cdot 3 + 1 = 8$

$(2,2) \to \frac{5}{2} \cdot 2 + \frac{3}{2} \cdot 2 + 1 = 9$

The optimum is obtained for $(x, y) = (1,1)$.

When $(x, y) = (1,1)$, $M = 1 - x + y = 1$, $M = \frac{3}{2} - \frac{x}{2} + \frac{y}{2} = 1$

When $(x, y) = (1,1)$, we have:

$m = \max\left\{\frac{3}{2} - 1 + \frac{1}{2}, 1 - 1\right\} = 1$ and $M = \min\left\{1 - 1 + 1, 1 - \frac{1}{2} + \frac{1}{2}\right\} = 1 \Rightarrow z^* = 1$

$z \in [\overset{m}{1}, \overset{M}{1}]$

$z = 1$

$z^* = 1$

3.2 The Proposed Method

Although the minimum number of points given by equation (4) $\varepsilon = 10^{-5}$ and $p = 10^{-3}$ is $N = 11507$, we use this example the following values $N = 10^6$.

Using the RandSeach algorithm for the previous numerical example we consider:

i) $N = 10^6$; $M = \frac{N}{10}$; $K = \frac{M}{10}$;

ii) the leader's function to optimize $F(x, y, z) := 3x + y + z$;

iii) the follower's function to optimize $f(x, y, z) := 2x - y - 7z$.

We obtain the approximate solution of this example
$\min F := 5.039$, $\min f := -6.052$, $x = 1.004$, $y = 1.012$, $z = 1.005$

By comparing the solution obtained by enumeration vertices method with the approximate solution given by proposed method we find the following relative error

$$\text{err} := \sqrt{\frac{\sum_{i=2}^{4} \left|s_{1,i} - S_{i-1}\right|^2}{\sum_{i=1}^{3} \left|S_i\right|^2}}$$

$\text{err} := 0.0085$

which leads us to affirm that the proposed method is "good"!

CONCLUSIONS

From the list of references can be seen that the BLP problem is still actual. The simplicity of the proposed approach would recommend it for practical applications. The only problem would be the finding an efficient algorithm for the numerical simulation of uniformly distributed points in the *D*. In the considered numeric example, we can be observed that the approximate solution is sufficiently "good", the relative error being of the 10^{-3} order.

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The work described has not been published previously and if accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright-holder.

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