SOME RESULTS ACCORDING THE INTERLACING THE ZEROS OF A FUNCTION

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Abstract: *The goal of this article is to present some results concerning Markov interlacing property of zeros for some kind of polynomials. We first present an equivalent lemma referring to T-systems from Borislav Bojanov's point of view and some extensions to more general classes of functions studied by Losko Milev and Nikola Naidenov. As a corollary we obtain that Markov's lemma holds true also in the case of logarithmic polynomials* $\sum_{k=0}^{n} a_k \ln^k x$.

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1. INTRODUCTION

During his research, the professor Borislav Bojanov was preoccupied to study and to bring new results referring Markov's lemma. He established an equivalent lemma concerning Chebyshev-systems in [1].

First of all, we give the definition of Chebyshev-systems.

Definition 1 [1] *Let* $\bar{\varphi} := \{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}$ *be an arbitrary system of continuous functions on the interval* $[a, b]$. *The system* $\bar{\varphi}$ *is a **Chebyshev-system** (or briefly **T-system**) on* $[a, b]$, *if*

$$det \begin{bmatrix} \varphi_1(t_1) & \cdots & \varphi_n(t_1) \\ \vdots & \ddots & \vdots \\ \varphi_1(t_n) & \cdots & \varphi_n(t_n) \end{bmatrix} \neq 0 \tag{1}$$

for each $t = (t_1, t_2, \dots, t_n)$ *such that* $a \le t_1 < t_2 < \dots, < t_n \le b$. *Assume in addition, that the functions* $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_{n-1}(x)\}$ *also constitute a T-system on* $[a, b]$.

We shall now consider the algebraic polynomials of degree at least equal to n.

Lemma of Markov [5] *Let* $p(x) = (x - x_1) \dots (x - x_n)$ *and* $q(x) = (x - y_1) \dots (x - y_n)$ *two polynomials that have the zeros* $x_1 < \dots < x_n$, *respectively,* $y_1 < \dots < y_n$ *that satisfy the interlacing conditions*

$$x_1 \le y_1 \le x_2 \le y_2 \le \dots \le x_n \le y_n \tag{2}$$

Then the zeros $t_1 \le t_2 \le \dots \le t_{n-1}$ *of* $p'(x)$ *and the zeros* $\tau_1 \le \tau_2 \le \dots \le \tau_{n-1}$ *of* $q'(x)$ *are interlacing too, i.e.*

$$t_1 \le \tau_1 \le t_2 \le \tau_2 \le \dots \le t_{n-1} \le \tau_{n-1} \tag{3}$$

Moreover, if $x_1 < \dots < x_n$ *and at least one of the inequalities* $x_i \le y_i$, $i = 1, \dots, n$ *is strict, then all inequalities in* (3) *are strict.*

Remark 1. The lemma of Markov is called also ***Markov interlacing property***.

2. EQUIVALENT RESULTS

In the paper [5], Lozko Milev and Nikola Naidenov formulated a condition (denoted by (**P**)), such that if a T-system satisfies (**P**), then there occurs Markov's lemma:

We consider now, $\{u_0, u_1, \dots, u_n\}$ be a T-system on $\mathbb{R}$, with $u_i \in C^n(\mathbb{R})$, for $i = 0,1,\dots,n$, and every non-zero of polynomial $u = \sum_{i=1}^{n} a_i u_1$, where $(a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ have at most n real zeros. We establish

$U_n := span\{u_0, u_1, \dots, u_n\}$,

and

$X = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 < \dots < x_n$.

Being $\bar{x} \in X$ a given point, we define

$$f(\bar{x}, t) := \begin{bmatrix} u_0(t) & \dots & u_n(t) \\ u_0(x_1) & \dots & u_n(x_1) \\ \vdots & \ddots & \vdots \\ u_0(x_n) & \dots & u_n(x_n) \end{bmatrix}. \tag{4}$$

Obviously, $f(\bar{x}, t)$ is a polynomial from U_n, which has zeros $x_1, x_2, \dots, x_n$. Note that if $g \in U_n$ is any other polynomial having the same zeros, then there exists a constant C such that $g(t) = Cf(\bar{x}, t)$. In general, we shall say that $f \in U_n$ is an *oscillating polynomial* if it has n distinct real zeros.

Applying Rolle's theorem to a polynomial $f(\bar{x}, t) \in U_n$, we observe that $f'(\bar{x}, t)$ admits at least one zero in each of the intervals $(x_i, x_{i+1}), i = 1, 2, \dots, n-1$. We shall suppose that the system U_n has the following property (**P**):

(**P**) *There exist numbers δ_0 and δ_n in $\{0,1\}$ such that for every $\bar{x} = (x_1, x_2, \dots, x_n) \in X, f'(\bar{x}, t)$ admit exactly:*

- *δ_0 zeros in $(-\infty, x_1)$;*
- *one zero in each of the interval $(x_i, x_{i+1}), i = 1, 2, \dots, n-1$;*
- *δ_n zeros in $(x_n, +\infty)$.*

In the following, we introduce the set of index $J(U_n) \subset \{0, 1, \dots, n\}$, which corresponds to the zeros of $f'(\bar{x}, t)$. The definition of (U_n) is as follows: *the set $\{1, \dots, n-1\}$ is contained in $J(U_n)$ and if $\delta_i = 1$ for some $i \in \{o, n\}$ then $i \in J(U_n)$.*

For the results that will be presented, we need to introduce the following theorem, which relates to the T-general systems which have the property (**P**).

Theorem 1 [5] *Assume that $U_n := span\{u_0, u_1, \dots, u_n\}$ is a T-system on the real line, which satisfy the property (**P**). Let $\bar{x} = (x_1, x_2, \dots, x_n)$ and $\bar{y} = (y_1, y_2, \dots, y_n)$ be two arrays from X, that are interlacing, in the following order*

$$x_1 \le y_1 \le x_2 \le y_2 \le \dots \le x_n \le y_n \tag{2}$$

Then the zeros $\{t_i\}_{i \in J(U_n)}$ of $f'(\bar{x}, t)$ and the zeros $\{\tau_i\}_{i \in J(U_n)}$ of $f'(\bar{y}, t)$ are interlacing in the same order:

$$t_m \le \tau_m \le t_{m+1} \le \tau_{m+1} \le \dots \le t_M \le \tau_M, \tag{5}$$

where $m := min\, J(U_n), M := max\, J(U_n)$. Even more, if $\bar{x} \neq \bar{y}$, then all the inequalities from (5) are strict.

Lemma 1 [5] *Let $\{u_0, u_1, \dots, u_n\}$ be a T-system on $\mathbb{R}$ and let f and g be two oscillating polynomials from $U_n := span\{u_0, u_1, \dots, u_n\}$ with the zeros $\bar{x} \in X$ and respectively, $\bar{y} \in X$. If $\bar{x}$ and $\bar{y}$ are interlacing and $\bar{x} \neq \bar{y}$, then f' and g' cannot have any common zero.*

Lemma 2 [1] *Every zero η of the derivative of an algebraic polynomial $p(x) = (x - x_1) \dots (x - x_n)$ is an increasing function x_k from the domain $x_1 < \dots < x_n$.*

3. MARKOV'S INTERLACING PROPERTY APPLIED ON OTHER TYPE OF POLYNOMIALS AND MAIN RESULTS

One of the natural objectives of mathematicians is to extend the Markov interlacing property on several general classes of functions. In researches that have been performed recently, Lozko Milev and Nikola Naidenov have established that Markov interlacing property of zeros occurs also in the case of exponential polynomials of the form $\sum_{i=0}^{n} b_i e^{\alpha_i X}$ and in the case of linear combinations of the Gaussian kernels (normal) $\sum_{i=0}^{n} b_i e^{-(X-\beta_i)^2}$ in the paper [5]. Other authors have demonstrated that Markov's lemma is maintained also in the case of orthogonal polynomial functions, trigonometric polynomial functions, etc.

We consider now the real numbers $\alpha_0 < \alpha_1 < \dots < \alpha_n$ and we define

$$V_n(\bar{\alpha}) := span\{e^{\alpha_0 x}, e^{\alpha_1 x}, \dots, e^{\alpha_n x}\}. \tag{6}$$

It is known the fact that $V_n(\bar{\alpha})$ is a T-system on $(-\infty, +\infty)$. Let

$$J(\bar{\alpha}) := \begin{cases} \{0, 1, \dots, n-1\}, \alpha_0 > 0; \\ \{1, 2, \dots, n\}, \alpha_n < 0; \\ \{1, 2, \dots, n-1\}, \alpha_0 \le 0 \le \alpha_n. \end{cases} \tag{7}$$

In the following theorem, we establish the Markov's interlacing property for the system $V_n(\bar{\alpha})$.

Theorem 2 [5] *Let f and g be two oscillating polynomials from $V_n(\bar{\alpha})$ that admit the zeros $\bar{x} \in X$ and $\bar{y} \in X$, respectively satisfy the inequalities (2). Then the zeros $\{t_i\}_{i \in J(\bar{\alpha})}$ of f' and zeros $\{\tau_i\}_{i \in J(\bar{\alpha})}$ of g' are interlacing in the same order:*

$$t_i \le \tau_i \le t_{i+1} \le \tau_{i+1}, \text{ for } i, i+1 \in J(\bar{\alpha}). \tag{8}$$

More, if $\bar{x} \neq \bar{y}$, then inequalities from (8) are strict.

In addition, if $\alpha_0 < \alpha_1 < \dots < \alpha_n < 0$ then for every natural number k, the zeros $\left\{t_i^{(k)}\right\}_{i=1}^{n}$ of $f^{(k)}$ and zeros $\left\{\tau_i^{(k)}\right\}_{i=1}^{n}$ of $g^{(k)}$ are interlacing also:

$$t_1^{(k)} \le \tau_1^{(k)} \le t_2^{(k)} \le \tau_2^{(k)} \le \dots \le t_n^{(k)} \le \tau_n^{(k)}.$$

A similar enunciation is true also for $0 < \alpha_0 < \alpha_1 < \dots < \alpha_n$.

Proof.

Consult [5] p. 674.

Remark 2. We will omit the proofs of the presented theory, they can be consulted in referred papers of each theorem/lemma/corollary.

Milev Lozko with Nikola Naidenov described the interlacing property of zeros for linear combinations of Gaussian kernels, in the following corollary:

Corollary 1 [5] *Let f and g be two oscillating polynomials of the form $\sum_{i=0}^{n} b_i e^{-(X-\beta_i)^2}$, $(\beta_0 < \dots < \beta_n)$ which admit the zeros $\bar{x}$, respectively $\bar{y}$. We consider that $\bar{x}$ and $\bar{y}$ are interlacing in the order of relation (1). Then the zeros $\{t_i\}_{i=0}^{n}$ of f' and the zeros $\{\tau_i\}_{i=0}^{n}$ of g' are interlacing in the same order. More, if $\bar{x} \neq \bar{y}$, then the interlacing is strict.*

Proof.

Consult [5] p. 676.

The author aims is to show that the Markov's interlacing property of zeros takes place in the case of logarithmic polynomials, by the following corollary:

Corollary 2 *Let f and g be two logarithmic polynomials of the form $\sum_{k=0}^{n} a_k \ln^k x$ that admit the zeros $\bar{x} = (x_1, \dots, x_n)$, respectively $\bar{y} = (y_1, \dots, y_n)$. We assume that $\bar{x}$ and $\bar{y}$ are interlacing in the order of relation (2). Then the zeros $\{t_i\}_{i=0}^{n}$ of f' and the zeros $\{\tau_i\}_{i=0}^{n}$ of g' are interlacing in the same order. Even more, if $\bar{x} \neq \bar{y}$, then the interlacing is strict.*

Proof.

First, we need to emphasize that the polynomial logarithmic with the form $\sum_{k=0}^{n} a_k \ln^k x$ is equivalent with the polynomial $\prod_{k=1}^{n}(\ln x - x_k)$. Therefore, since this is a product composed of increasing factors with functions (regardless of the actual values of x_k), we obtain a polynomial function admitting n real solutions, like:

- ✓ to the function $f: (0; +\infty) \to \mathbb{R}, f(x) = \prod_{k=1}^{n}(\ln x - x_k)$, if we attach the equation $f(x) = 0$, we will have

$$(\ln x - x_1)(\ln x - x_2) \dots (\ln x - x_n) = 0 \Rightarrow \begin{cases} (\ln x - x_1) = 0 \\ (\ln x - x_2) = 0 \\ \dots \\ (\ln x - x_n) = 0 \end{cases} \Rightarrow \tag{9}$$

$$\Rightarrow \begin{cases} \ln x = \ln e^{x_1} \\ \ln x = \ln e^{x_2} \\ \dots \\ \ln x = \ln e^{x_n} \\ \text{fc. ln is injective} \end{cases} \Rightarrow \bar{x} = (e^{x_1}, e^{x_2}, \dots, e^{x_n})$$

therefore the polynomial solutions of f are of the form $\bar{x} = (e^{x_1}, e^{x_2}, \dots, e^{x_n})$, and those of g are of the form $\bar{y} = (e^{y_1}, e^{y_2}, \dots, e^{y_n})$. We insist to underline that these solutions are unique, that is resulting from the fact that this logarithmic function is injective over the entire domain of definition. Here, we shall attach the graph of the function $f: (0; +\infty) \to \mathbb{R}, f(x) = \prod_{k=1}^{n}(\ln x - x_k)$, to outline the idea that Lemma 2 is carried also in the present case (by changing the variable $x \longrightarrow \ln x$), i.e. algebraic solutions $\bar{x}$ and $\bar{y}$ of polynomials f, respectively g are increasing functions from the domain $x_1 < \dots < x_n$.

$$f(x) = \prod_{k=1}^{n}(\ln x - x_k) \tag{10}$$

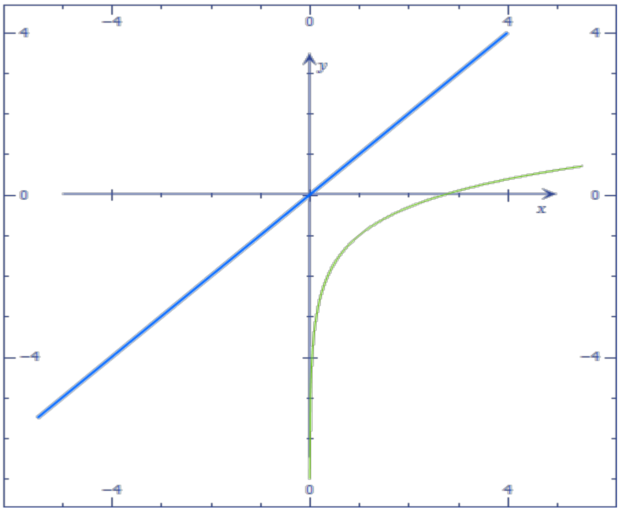

FIG. 1. The graph of function $f(x)=\prod_{k=1}^{n}(\ln x - x_k)$

We now consider, $\{\alpha_0 \ln^0 x, \alpha_1 \ln^1 x, \alpha_2 \ln^2 x, \dots, \alpha_n \ln^n x\}$ a T-system on $\mathbb{R}$, with $\ln^i x \in C^n(\mathbb{R})$, for $i=0,1,\dots,n$, and every non-zero of polynomial $\sum_{i=0}^{n} a_i \ln^i x$, where $(a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ it has at most n real zeros. We set

$$U_n := span\{\alpha_0 \ln^0 x, \alpha_1 \ln^1 x, \alpha_2 \ln^2 x, \dots, \alpha_n \ln^n x\} \tag{11}$$

and

$$X=\{(e^{x_1}, e^{x_2}, \dots, e^{x_n}) \in \mathbb{R}^n : e^{x_1} < e^{x_2} < \dots < e^{x_n};\ x_1 < \cdots < x_n\}. \tag{12}$$

Given a point $\bar{x} \in X$, we define

$$f(\bar{x}, t) := \begin{bmatrix} \ln^0(t) & \dots & \ln^n(t) \\ \ln^0(e^{x_1}) & \dots & \ln^n(e^{x_1}) \\ \vdots & \ddots & \vdots \\ \ln^0(e^{x_n}) & \dots & \ln^n(e^{x_n}) \end{bmatrix}. \tag{13}$$

Evidently, $f(\bar{x}, t)$ is a polynomial from U_n, which has the zeros $e^{x_1}, e^{x_2}, \dots, e^{x_n}$. We emphasize that if $g \in U_n$ is another logarithmic polynomial that has the zeroes $\bar{y}=(e^{y_1}, e^{y_2}, \dots, e^{y_n})$. We have then $f \in U_n$ and $g \in U_n$ being two oscillating polynomials that have n real distinct zeros.
Applying Rolle's theorem to the polynomial $f(\bar{x}, t) \in U_n$, with

$$f(x)=\sum_{k=0}^{n} a_k \ln^k x \Leftrightarrow f(x)=\prod_{k=1}^{n}(\ln x - x_k) \Rightarrow$$
$$\Rightarrow f'(x)=\frac{1}{x}a_1+\frac{2}{x}a_2 \ln x+\frac{3}{x}a_3 \ln^2 x+\cdots+\frac{n}{x}a_n \ln^{n-1} x \Leftrightarrow f'(x)=$$
$$=\frac{1}{x}(\ln x - x_2)(\ln x - x_3)\dots(\ln x - x_n)$$
$$+\frac{1}{x}(\ln x - x_1)(\ln x - x_3)\dots(\ln x - x_n)+\cdots$$
$$+\frac{1}{x}(\ln x - x_1)(\ln x - x_2)\dots(\ln x - x_{n-2})(\ln x - x_n)$$
$$+\frac{1}{x}(\ln x - x_1)(\ln x - x_2)\dots(\ln x - x_{n-2})(\ln x - x_{n-1}) \Rightarrow$$

$$\Rightarrow f'(x) = \frac{1}{x}\sum_{k=1}^{n} a_k \ln^{k-1} x \Leftrightarrow f'(x) =$$
$$= \frac{1}{x}[(\ln x - x_2)(\ln x - x_3) \dots (\ln x - x_n)$$
$$+ (\ln x - x_1)(\ln x - x_3) \dots (\ln x - x_n) + \cdots$$
$$+ (\ln x - x_1)(\ln x - x_2) \dots (\ln x - x_{n-2})(\ln x - x_n)$$
$$+ (\ln x - x_1)(\ln x - x_2) \dots (\ln x - x_{n-2})(\ln x - x_{n-1})]$$

where $\frac{1}{x} \neq 0, \forall x \in (0, +\infty)$.

We can observe that the derivative $f'(\bar{x}, t)$ is another logarithmic polynomial function of degree $n-1$, which admits at least one zero in each of the intervals (x_i, x_{i+1}), $i = 1, 2, \dots, n-1$, on the form $e^{t_1}, e^{t_2}, \dots, e^{t_n}$, respectively $g'(\bar{y}, t)$ admits at least one zero in each of the intervals $(y_i, y_{i+1}), i = 1, 2, \dots, n-1$, on the form $e^{\tau_1}, e^{\tau_2}, \dots, e^{\tau_n}$; and the system U_n admits the property (**P**), therefore by theorem 1, it happens the interlacing of the zeros of $f'(\bar{x}, t)$ and $g'(\bar{y}, t)$ like this

$$e^{t_1} \leq e^{\tau_1} \leq e^{t_2} \leq e^{\tau_2} \leq \cdots \leq e^{t_n} \leq e^{\tau_n}. \tag{14}$$

We must point out that, according to Lemma 1, f' and g' do not have any common zero.

We want now to justify the strictly intercalation, as follows: because $\bar{x} \neq \bar{y}$, thanks to the exponential function to be strictly monotone, respectively injective; it follows therefore that the interlacing is strict

$$e^{t_1} < e^{\tau_1} < e^{t_2} < e^{\tau_2} < \cdots < e^{t_n} < e^{\tau_n} \tag{15}$$

which finish the proof of the corollary.

Corollary 3 *Let f and g be two logarithmic polynomials of the form $\sum_{k=0}^{n} a_k \ln^k x$ that admit the zeros $\bar{x} = (x_1, \dots, x_n)$, respectively $\bar{y} = (y_1, \dots, y_n)$. We assume that $\bar{x}$ and $\bar{y}$ are interlacing in the order of relation (1). Then the zeros $\{t_i\}_{i=0}^{n}$ of derivative of order (k) of f, noted by $f^{(k)}$ and the zeros $\{\tau_i\}_{i=0}^{n}$ of derivative of order (k) of g, noted by $g^{(k)}$ are interlacing in the same order. Even more, if $\bar{x} \neq \bar{y}$, then the interlacing is strict.*

Proof.

The proof can be done by induction, with the particular case $k = 1$ demonstrated in corollary 2. (with $k = 1$ we discuss about the first derivative of the polynomials f and g, i.e. f' respectively g').

CONCLUSIONS

The studies of Borislav Bojanov referring the lemma of Markov, have applications in algebra, approximation theory, statistics, etc. The author aims to expand the two corollaries presented aforesaid, in the probability theory. Maybe it can be find a connection with statistical distributions, like the logarithmic – exponential distribution.

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