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ABOUT A PAIR LINEAR POSITIVE OPERATORS ASSOCIATED WITH BLEIMANN-BUTZER-HAHN OPERATOR

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Abstract. *We deal in this paper with an estimation of the difference between Bleimann-Butzer-Hahn operator and its associated operator defined according to a general method of construction of linear positive operator.*

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1. INTRODUCTION

In our paper [4] we defined and studied the approximation properties of a new linear positive operator associated with Bleimann-Butzer-Hahn operator obtained according to a general method of construction of linear positive operators.

Indeed, this method means to associate to the operator $P_n : \mathcal{L} \to \mathcal{F}(I)$ defined as

$$P_n(f;x) = \sum_{k=0}^{n} h_{n,\, k}(x)\ f\left(x_{n,\, k}\right),\ f \in \mathcal{L} \tag{1.1}$$

a linear positive operator of the form

$$L_n(f;x) = \sum_{k=0}^{n} h_{n,\, k}(x)\ \nu_{n,\, k}(f),\ x \in I,\ f \in \mathcal{L}, \tag{1.2}$$

where $h_{n,\, k} \in C_B(I),\ h_{n,\, k} \geq 0$ so that $\sum_{k=0}^{n} h_{n,\, k} = 1$, exists $x_{n,\, k} \in I$ the barycenter of a: $\mu_{n,\, k}$ probability Borel measures on I, $n \geq 1,\ k = \overline{0,n}$ i.e. $x_{n,\, k} = \int_I t\ d\mu_{n,\, k}(t)$ and $\nu_{n,\, k}(f) = \int_I f(t)\ d\mu_{n,\, k}(t),\ f \in \mathcal{L}.$

We consider that, $\mathcal{L}$ is the common set of real measurable bounded functions on I for which $P_n f,\ L_n f,\ \nu_{n,\, k}(f)$ are well defined and $\mathcal{F}(I)$ is the space of all real valued functions defined on I. As usual, $e_i(x) = x^i,\ i = 0,\ 1,\ 2,\ x \in I$ denote the test monomial functions.

For the pair of linear positive operators (P_n, L_n) it is true the next result [5]:

Theorem 1.1. *If* $(L_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$, *are two sequences of linear positive operators defined as* (1.1) *respectively* (1.2) *for* $f \in C^2{}_B(I) \subset \mathcal{L}$, *then for* $x \in I$ *we have the estimation*

$$\left|L_n(f;x)-P_n(f;x)\right|\le\frac{\|f''\|}{2}\cdot\sum_{k\ge0}h_{n,k}(x)\left[\nu_{n,k}(e_2)-\left(\nu_{n,k}(e_1)\right)^2\right].$$

So, we consider that $P_n:C_B[0,\infty)\to C_B[0,\infty)$ is the Bleimann-Butzer-Hahn operator [1], [2], [3], [7], defined as

$$P_n(f;x)=(1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k f\left(\frac{k}{n-k+1}\right),\quad f\in C_B[0,+\infty),\ x\ge0,\ n\in N, \tag{1.3}$$

and its associated linear positive operator according to the general method of construction is the new linear positive operator $L_n:C_B[0,\infty)\to C_B[0,\infty)$ defined in [4] as

$$L_n(f;x)=\frac{1}{(1+x)^n}f(0)+\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\cdot\frac{1}{B(k,\ n-k+2)}\int_0^\infty f(t)\frac{t^{k-1}}{(1+t)^{n+2}}dt+$$

$$+\left(\frac{x}{1+x}\right)^n f(n),\ x\ge0,\ f\in C_B[0,+\infty)\ , \tag{1.4}$$

with $B(a,\ b)=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt,\ a>0,\ b>0$

the Inverse-Beta function.

2. AN ESTIMATION ON THE DIFFERENCE $|L_n f-P_n f|$

Using the theorem 1.1 we give an estimation of the difference $|L_n f-P_n f|$. So,

$$\left|L_n(f;x)-P_n(f;x)\right|\le\frac{\|f''\|}{2}\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}\cdot$$

$$\cdot\left[\int_0^\infty\frac{t^2}{B(k,\ n-k+2)}\cdot\frac{t^{k-1}}{(1+t)^{n+2}}dt-\left(\frac{k}{n-k+1}\right)^2\right]$$

$$=\frac{\|f''\|}{2}\sum_{k=1}^{n-1}\binom{n}{k}\frac{x^k}{(1+x)^n}$$

$$\left[\frac{B(k+2,\ n-k)}{B(k,\ n-k+2)}-\frac{k^2}{(n-k+1)^2}\right]=$$

$$=\frac{\|f''\|}{2}\sum_{k=1}^{n-1}\binom{n+1}{k}k\frac{x^k}{(1+x)^n}\left[\frac{1}{n-k}-\frac{1}{n-k+1}\right]=$$

$$=\frac{\|f''\|}{2}\sum_{k=1}^{n-1}\binom{n+1}{k-1}\frac{x^k}{(1+x)^n}\left[\frac{2}{n-k}-\frac{1}{n-k+1}\right]=$$

$$=\frac{\|f''\|}{2}\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{x^{j+1}}{(1+x)^n}\left[\frac{2}{n-j-1}-\frac{1}{n-j}\right]=$$

$$=\frac{\|f''\|}{2}\left[R-\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{1}{n-j}\cdot\frac{x^{j+1}}{(1+x)^n}\right] \tag{1.5}$$

with

$$R=\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{2}{n-j-1}\cdot\frac{x^{j+1}}{(1+x)^n}\le\frac{8x(1+x)^2}{n+2}$$

(see [4]). (1.6)

Since,

$$\frac{1}{n-j}\le\frac{2}{n-j+2},\quad 0\le j\le n-2,\ n\ge2$$

we have for the second term of (1.5) that

$$\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{1}{n-j}\cdot\frac{x^{j+1}}{(1+x)^n}\le2\sum_{j=0}^{n-2}\binom{n+1}{j}\cdot$$

$$\cdot\frac{1}{n-j+2}\cdot\frac{x^{j+1}}{(1+x)^n}\le$$

$$\le2x\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{1}{n-j+2}\cdot\frac{x^j}{(1+x)^n}$$

$$\leq 2x(1+x)\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{1}{n-j+2}\cdot\left(\frac{x}{1+x}\right)^{j}\cdot\left(1-\frac{x}{1+x}\right)^{n+1-j}=2x(1+x)E\left[\frac{1}{n-U+2}\right] \quad (1.7)$$

Together with a result of Chao and Strawdermann [6, (3.4)] we have for the mean value of the random variable $\frac{1}{n+2-U}$ when $n+1-U$ has a Bernoulli distribution with parameters $n+1$ and $q=1-p=\frac{1}{1+x}$, that

$$E\left[\frac{1}{n+2-U}\right]=E\left[\frac{1}{1+(n+1-U)}\right]=\frac{1-p^{n+2}}{(n+2)q}<\frac{1}{(n+2)q}=\frac{1+x}{n+2} \quad (1.8)$$

So, using (1.5) with (1.6), (1.7), (1.8) we obtain

$$|L_n(f;x)-P_n(f;x)|\leq\frac{\|f''\|}{2}\left[\frac{8x(1+x)^2}{n+2}-\frac{2x(1+x)^2}{n+2}\right],$$

$$|L_n(f;x)-P_n(f;x)|\leq\frac{3x(1+x)^2}{n+2}\|f''\|.$$

Theorem 2.1. *For* $n\geq 2,\ \ x\in[0,\infty),\ \ f\in C^2{}_B[0,\infty)$ *we have to relative to the pair of the operators* (1.3) *and* (1.4)

$$|L_n(f;x)-P_n(f;x)|<\frac{3x(1+x)^2}{n+2}\|f''\| .$$

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