"HENRI COANDA"
AIR FORCE ACADEMY
ROMANIA

"GENERAL M.R. STEFANIK"
ARMED FORCES ACADEMY
SLOVAK REPUBLIC

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INTEGRAL EQUATION FOR ELASTIC CONTACT BETWEEN A PROJECTILE - RIGID BODY AND A TARGET - ELASTIC SURFACE

IVĂNICĂ, Mihai

Military Technical Academy, Bucharest, Romania

Abstract: *This paper theoretically treats, using a complex mathematical model, the elastic contact between bodies, taking into consideration a rigid body named projectile and an elastic one with a plane surface contact (target surface). After the contact has been produced and the projectile deforms the elastic surface in a certain unknown depth, the initial considered contact point will be transformed in a contact domain, and the stresses appeared on this contact domain will equilibrate the system forces which acts over the projectile. From this point of view, the paper offers solutions both for the contact domain form and integral equation for elastic contact.*

Keywords: *elastic contact, rigid body, elastic surface.*

1. INTRODUCTION

Is assumed, in all that follows, that one of the bodies in contact is an elastic semispace, and the other one is a hard body called projectile.

It is considered semispace $z \geq 0$, subjected to a normal charge $p(\xi,\eta)$ distributed over a finite domain ***D*** situated at its boundary $z = 0$. Current point coordinates in ***D*** are ξ and η (fig. no 1). Newmann's problem for elastic semispace consists in finding a solution which describes the displacement field inside the semispace in the following limit conditions:

$$\begin{cases} \sigma_z\big|_{z=0} = \begin{cases} -p(\xi,\eta) \_\_\_ (\xi,\eta)\in D \\ 0 \_\_\_\_\_\_\_\_ (\xi,\eta)\notin D \end{cases} \\ \tau_{zx}\big|_{z=0} = \tau_{zy}\big|_{z=0} = 0 \_\_\_\_\_\_\_\_ \end{cases} \quad (1)$$

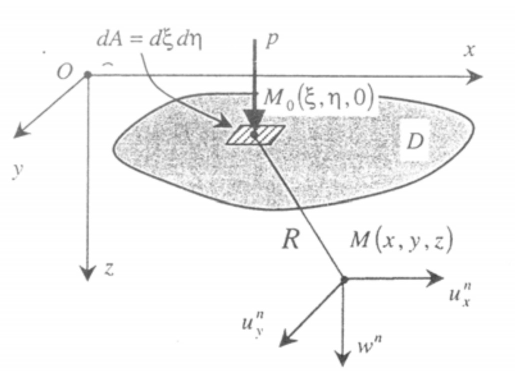

Fig. no 1

The solution for this problem can be obtained either from the classic Boussinesq's problem, or Neuberg-Papkovici representation.

2. THE TEXT OF THE PAPER

2.1 Boussinesq's problem for elastic semispace. Boussinesq's problem for elastic semispace consists in describing the action of a force concentrated in a point on the boundary of an elastic semispace, normal to this boundary. Usually, this point is chosen as being the origin of an orthogonal reference system with axes ***Ox*** and ***Oy*** in the semispace boundary plan. ***M(x,y,z)*** point displacements inside the semispace under the ***P*** force action are:

$$\begin{cases} u_x = \dfrac{1+\nu}{2\pi E} P\left[\dfrac{xz}{R'^3} - (1-2\nu)\dfrac{x}{R'(R'+z)}\right] \\ u_y = \dfrac{1+\nu}{2\pi E} P\left[\dfrac{yz}{R'^3} - (1-2\nu)\dfrac{y}{R'(R'+z)}\right] \\ w = \dfrac{1+\nu}{2\pi E} P\left[\dfrac{z^2}{R'^3} + 2(1-\nu)\dfrac{1}{R'}\right] \end{cases} \quad (2)$$

with $R' = \sqrt{x^2 + y^2 + z^2}$ (3)

Replacing in this formula the concentrated charge ***P*** with $p(\xi,\eta)d\xi d\eta$, x with $\xi - x$ and y with $\eta - y$, respectively distance R' with:

$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2} \quad (4)$$

with integration on D will lead to ***M(x,y,z)*** point displacement as in fig. no 1

Neuberg-Papkovici representation assumes seeking displacement vector as:

$$\overline{U} = \overline{B} - \frac{1}{4(1-\nu)} grad\left(\overline{r}\cdot\overline{B} + B_0\right) \quad (5)$$

where $\overline{B}$ is a harmonic vector function $(\Delta\overline{B} = 0)$, ***B₀*** is a scalar vector function ($\Delta B_0 = 0$), and $\overline{r}$ is the position vector.

In both situations the solution is written using potential simple layer functions (harmonic functions in ***D***),

$$\Omega(x,y,z) = -\frac{1+\nu}{2\pi E}\iint_D p(\xi,\eta)\frac{1}{R}d\xi d\eta$$
$$\omega(x,y,z) = -\frac{1+\nu}{2\pi E}\iint_D p(\xi,\eta)\ln(R+z)d\xi d\eta \quad (6)$$

which allows the characteristic

$$\Omega(x,y,z) = \frac{\partial\omega}{\partial z}(x,y,z) \quad (7)$$

This solution is:

$$\begin{cases} u_x^n = z\dfrac{\partial\Omega}{\partial x} + (1-2\nu)\dfrac{\partial\omega}{\partial x} \\ u_y^n = z\dfrac{\partial\Omega}{\partial y} + (1-2\nu)\dfrac{\partial\omega}{\partial y} \\ w^n = z\dfrac{\partial\Omega}{\partial z} - 2(1-\nu)\Omega \end{cases} \quad (8)$$

A more compact representation of those equations can be made by introducing complex displacement:

$$u_c^n = u_x^n + iu_y^n \quad (9)$$

and differential operator $\Lambda = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$ (10)

This operator satisfies the relation:

$$\Lambda\overline{\Lambda} = \overline{\Lambda}\Lambda = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (11)$$

being its complex adjoint. With those notations equations (8) become:

$$\begin{cases} u_c^n = -\dfrac{1+\nu}{2\pi E}\left[z\Lambda\dfrac{\partial I_n}{\partial z} + (1-2\nu)\Lambda I_n\right] \\ w^n = -\dfrac{1+\nu}{2\pi E}\left[z\dfrac{\partial^2 I_n}{\partial z^2} - 2(1-\nu)\dfrac{\partial I_n}{\partial z}\right] \end{cases} \quad (12)$$

where

$$I_n(x,y,z) = \iint_D p(\xi,\eta)\ln(R+z)d\xi d\eta \quad (13)$$

2.2 Paraboloidal projectile case with central action.

The classic theory of elastic contact admits that the surface projectile body is curved, the contact with semispace is produced in an initial considered point ***O*** (fig. no 2) in which two cartesian reference systems are built: ***Oxyz*** attached to the elastic semispace and ***OXYZ*** attached to the projectile. If $Z = \varphi(X,Y)$ is the boundary projectile equation, with φ a C2 class function on parts, where point ***O*** is considered to be an elliptic one of this surface. Also, the plan $OXY \equiv Oxy$ is a tangent plan in point ***O*** at the projectile boundary:

$$\varphi(0,0) = \frac{\partial\varphi}{\partial X}(0,0) = \frac{\partial\varphi}{\partial Y}(0,0) = 0 \quad (14)$$

Is assumed that the projectile is subjected to the action of a forces system, reducible to a force and a couple (because the projectile is a rigid, hard body), which allow it penetrate in the elastic semispace. Projectile – semi space system equilibrium is established when the projectile will be penetrated to a certain depth (unknown) in semispace, when the initial considered contact point will be transformed in a contact domain (unknown) on which the projectile comes in touch with the deformed boundary of the semispace and when the stresses (also unknown) on this domain will reach the equilibrium of the forces applied to the projectile. In the absence of friction it can be assumed that the projectile is subjected to a vertical force ***P***, which support go through the point (ξ_0, η_0) from the plan $z = 0$. The displacements and deformations that occur are assumed to be small enough to allow using the linear theory. In this way the limit conditions can be expressed on the nondeformed boundary of the semispace [2].

Thus, although the real contact domain has the projectile's shape, contact domain is referred as being the ***D*** domain of plan $z = 0$ which after deformations gets in contact, point by point, with boundary projectile. So the equilibrium conditions for projectile are:

$$\iint_D p(\xi,\eta)d\xi d\eta = P\,,\ \iint_D \xi p(\xi,\eta)d\xi d\eta = \xi_0 P\,,$$
$$\iint_D \eta p(\xi,\eta)d\xi d\eta = \eta_0 P \qquad (15)$$

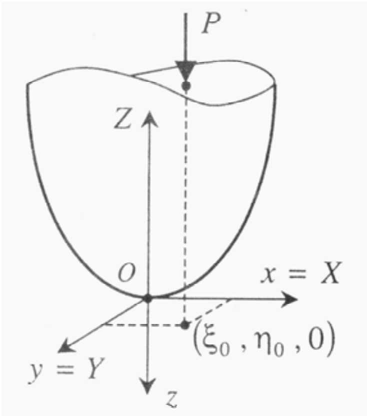

Fig. no 2

The limit conditions on the semispace boundary are given by (1), being specific to a contact surface without friction. Not considering the tangential displacement u_x^n and u_y^n in relation to ***w*** displacement on the contact domain from fig. no 3 [2], the following equation is obtained:

$$w = \delta - \varphi(x,y), (x,y) \in D \qquad (16)$$

which combined with equation $(12)_2$ for $z = 0$ leads to:

$$\frac{1-\nu^2}{\pi E}\iint_D \frac{p(\xi,\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}d\xi d\eta = \delta - \varphi(x,y),\ (x,y)\in D \qquad (17)$$

called integral equation for elastic contact. Distance δ is the maxim depth where the projectile penetrates semispace and is called interpenetration. Equation (17) has an extreme complexity having $p(\xi,\eta), D, \delta$ unknown.

In paraboloidal projectile case with central action, characterized by:

$$\varphi(X,Y) = \frac{1}{2}\left(\frac{X^2}{\rho'} + \frac{Y^2}{\rho''}\right) \qquad (18)$$

equation (17) is written

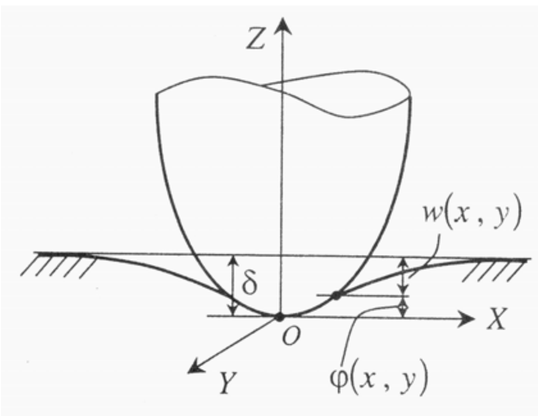

Fig. no 3

$$\frac{1-\nu^2}{\pi E}\iint_D \frac{p(\xi,\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}d\xi d\eta = \delta - \frac{1}{2}\left(\frac{x^2}{\rho'}+\frac{y^2}{\rho''}\right),\ (x,y)\in D \qquad (19)$$

at which is added the equilibrium condition $(15)_1$. In equations (18) and (19) ρ' and ρ'' are the main radius curves of projectile in ***O***, system axes ***Oxyz*** being oriented so that $\rho' > \rho''$.

3. CONCLUSIONS & ACKNOWLEDGMENT

The contact problem between a rigid paraboloidal body and an elastic semispace is of the most importance, because the paraboloidal projectile equation is valid at first approximation for projectile of any configuration in the near by of any elliptic point of its boundary. So the contact domain is of elliptic shape

$$D=\left\{(x,y)\left|\frac{x^2}{y^2}+\frac{y^2}{b^2}\le 1, b\le a\right.\right\} \qquad (20)$$

with ***a*** and ***b*** unknown semiaxes. The integral equation for elastic contact (19) is verified by pressure function

$$p(\xi,\eta)=\frac{3}{2}\frac{P}{\pi ab}\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}} \qquad (21)$$

and the interpenetration projectile in semispace is:

$$\delta=\frac{3}{2}P\frac{1-\nu^2}{\pi E}\frac{K(k)}{a} \qquad (22)$$

where

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\frac{\pi}{2}\left\{1+\sum_{p=1}^{\infty}\left[\frac{(2p-1)!!}{(2p)!!}\right]^2\cdot k^{2p}\right\},\ 0\le k<1 \qquad (23)$$

K(k) is elliptical second kind integral of Legendre [1] with $k=\sqrt{1-\frac{b^2}{a^2}}$ ellipse eccentricity.

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